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Approximation by Complex Operators on Unit Disk

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Abstract

This paper studies the properties of complex Picard's approximation, we use continuous complex operators to examine how well they approximate a function in L_p- quasi normed spaces in terms of the second-degree smoothness modulus. Using the definition $L_p(\mathcal{D})$, we obtain some approximation properties for complex singular integrals $\mathcal{K}_{\mathcal{G}}(f)(z)$.

Keywords: Complex Picard integrals; complex singular integrals; modulus of smoothness.

التقريب بواسطة المؤثر العقدي حول قرص الوحدة

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الخلاصة

الهدف من هذا البحث هو دراسة خصائص تقريب بيكارد المعقدة، حيث استخدمنا مؤثرات المعقدة المستمرة لفحص مدى نجاحها في تقريب دالة في الفضاءات شبه المعيارية L_p بدلالة معامل النعومة من الدرجة الثانية. باستخدام التعريف $L_p(\mathcal{D})$، نحن حصلنا على بعض الخصائص التقريبية للتكاملات المفردة المعقدة $\mathcal{K}_{\mathcal{G}}(f)(z)$
.

1. Introduction

Define $L_p(\mathcal{D}) = \left\{f : \mathcal{D} \longrightarrow \mathbb{C} : \|f\|_{L_p(\mathcal{D})} = \left(\int_{\mathcal{D}} |f|^p\right)^{\frac{1}{p}} < \infty\right\}$ where $0 < p < 1$, where $\mathcal{D} = \{z \in \mathbb{C}: |z| < 1\}$ is the open unit disk and $L_p(\overline{\mathcal{D}}) = \{f: \overline{\mathcal{D}} \longrightarrow \mathbb{C};\ f(0) = 0, f'(0) = 1\}$. Therefore, if $f \in L_p(\overline{\mathcal{D}})$, we have $f(z) = z + \sum_{\ell=2}^{\infty} b_\ell z^\ell, \forall\ z \in \mathcal{D}$.

Anastassiou & Gal [2] introduced & researched the complex singular integrals for $\mathcal{G} \in \mathbb{R}$ and $\mathcal{G} > 0$,

$$\mathcal{K}_{\mathcal{G}}(f)(z) = \frac{1}{2\mathcal{G}} \int_{-\infty}^{+\infty} f\left(ze^{i\mathcal{V}}\right) e^{-|\mathcal{V}|/\mathcal{G}} d\mathcal{V}, \qquad z \in \overline{\mathcal{D}},$$

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$$\mathcal{N}_{\mathcal{G}}(\mathfrak{f})(z) = \frac{\mathcal{G}}{\pi}\int_{-\pi}^{\pi} \frac{\mathfrak{f}(ze^{i\mathcal{V}})}{\mathcal{V}^2+\mathcal{G}^2} d\mathcal{V}, \qquad z \in \overline{\mathcal{D}},$$

$$\mathcal{N}_{\mathcal{G}}^{*}(\mathfrak{f})(z) = \frac{\mathcal{G}}{\pi}\int_{-\infty}^{+\infty} \frac{\mathfrak{f}(ze^{i\mathcal{V}})}{\mathcal{V}^2+\mathcal{G}^2} d\mathcal{V}, \qquad z \in \overline{\mathcal{D}},$$

$$\mathcal{X}_{\mathcal{G}}(\mathfrak{f})(z) = \frac{2\mathcal{G}^3}{\pi}\int_{-\infty}^{+\infty} \frac{\mathfrak{f}(ze^{i\mathcal{V}})}{(\mathcal{V}^2+\mathcal{G}^2)^2} d\mathcal{V}, \qquad z \in \overline{\mathcal{D}},$$

$$\mathcal{T}_{\mathcal{G}}(\mathfrak{f})(z) = \frac{1}{\sqrt{\pi\mathcal{G}}}\int_{-\pi}^{\pi} \mathfrak{f}(ze^{i\mathcal{V}}) e^{-\mathcal{V}^2/\mathcal{G}} d\mathcal{V}, \qquad z \in \overline{\mathcal{D}},$$

$$\mathcal{T}_{\mathcal{G}}^{*}(\mathfrak{f})(z) = \frac{1}{\sqrt{\pi\mathcal{G}}}\int_{-\infty}^{+\infty} \mathfrak{f}(ze^{-i\mathcal{V}}) e^{-\mathcal{V}^2/\mathcal{G}} d\mathcal{V}, \qquad z \in \overline{\mathcal{D}}. \qquad (1.1)$$

where $\overline{\mathcal{D}}$ is closed unit disk.

In this case, $\mathcal{N}_{\mathcal{G}}(\mathfrak{f})(z)$, $\mathcal{N}_{\mathcal{G}}^{*}(\mathfrak{f})(z)$ and $\mathcal{X}_{\mathcal{G}}(\mathfrak{f})(z)$ are said to be of Poiisson- Cauchy typee, $\mathcal{K}_{\mathcal{G}}(\mathfrak{f})(z)$ is said to be of Picard type and $\mathcal{T}_{\mathcal{G}}(\mathfrak{f})(z)$ and $\mathcal{T}_{\mathcal{G}}^{*}(\mathfrak{f})(z)$ are said to be of Gaus-Weiierstrass type.

In [1,3-9], classes of convolution complex polynomials were introduced, and some geometric properties such as the preservation of coefficients' bounds, bounded turn, positivity of real part, univalence were proved, starlikness and convexity.

The objective of this paper is to study geometric and approximation properties by using definition of $L_p(\mathcal{D})$.

Here we use the singular integral operator to study its approximation of functions in $L_p(\overline{\mathcal{D}})$ in terms of the second order modulus of smoothness.

2. Complex Picard integrals

In the following paper, we examine the properties of $\mathcal{K}_{\mathcal{G}}(\mathfrak{f})(z)$.

We start by explaining its approximation properties.

Theorem 2.1: If $\mathfrak{f} \in L_p(\overline{\mathcal{D}})$ $and\ \mathcal{G} > 0$, then

(i) $\mathcal{K}_{\mathcal{G}}(\mathfrak{f})(z)$ is continuous on $\overline{\mathcal{D}}$.

(ii) $\omega_1(\mathcal{K}_{\mathcal{G}}(\mathfrak{f});\beta)_{\overline{\mathcal{D}}} \le c(p)\frac{1}{n}\omega_1(\mathfrak{f};\beta)_{\overline{\mathcal{D}}}$, where $\omega_1(\mathfrak{f};\beta)_{\overline{\mathcal{D}}} = \sup_{|h|<\beta}\|\mathfrak{f}(z+h)-\mathfrak{f}(z)\|_{L_p(\mathcal{D})}, \beta > 0$.

(iii) $\left\|\mathcal{K}_{\mathcal{G}}(\mathfrak{f})(z)-\mathfrak{f}(z)\right\|_{L_p(\mathcal{D})} \le c(p)\psi\omega_3(\mathfrak{f};\mathcal{G})_{\mathcal{D}}, where$

$$\omega_3(\mathfrak{f};\mathcal{G})_{\mathcal{D}} = \left\|2\mathfrak{f}(ze^{i\mathcal{V}}) - 4\mathfrak{f}(z) + 2\mathfrak{f}(ze^{-i\mathcal{V}})\right\|_{L_p(\mathcal{D})}, |\mathcal{V}| \le \mathcal{G}. \quad (2.1)$$

Proof: (i) Let $z, z_0 \in \overline{\mathcal{D}}$, we obtain

$$\left\|\mathcal{K}_{\mathcal{G}}(\mathfrak{f})(z)-\mathcal{K}_{\mathcal{G}}(\mathfrak{f})(z_0)\right\|_{L_p(\mathcal{D})} = \left\|\frac{1}{2\mathcal{G}}\int_{-\infty}^{+\infty}\left|\mathfrak{f}(ze^{i\mathcal{V}})-\mathfrak{f}(z_0e^{i\mathcal{V}})\right|e^{-|\mathcal{V}|/\mathcal{G}}d\mathcal{V}\right\|_{L_p(\mathcal{D})}$$

$$\le \left\|\frac{1}{2\mathcal{G}}\int_{-\infty}^{+\infty}\left|\Delta_h^1\mathfrak{f}(ze^{i\mathcal{V}})\right|e^{-|\mathcal{V}|/\mathcal{G}}d\mathcal{V}\right\|_{L_p(\mathcal{D})}, h < \left|ze^{i\mathcal{V}} - z_0e^{i\mathcal{V}}\right|$$

$$\le \left\|\frac{1}{2\mathcal{G}}\sum_{j=1}^{\infty}\left|\Delta_h^1\mathfrak{f}(ze^{i\mathcal{V}_j})\frac{1}{n}\right|e^{-|\mathcal{V}_j|/\mathcal{G}}\left|\mathcal{V}_j-\mathcal{V}_{j+1}\right|\right\|_{L_p(\mathcal{D})}$$

$$\le c(p)e^{-|\mathcal{V}_j|/\mathcal{G}}\left|\mathcal{V}_j-\mathcal{V}_{j+1}\right|\frac{1}{n}\sum_{j=1}^{\infty}\left\|\frac{1}{2\mathcal{G}}\left|\Delta_h^1\mathfrak{f}(ze^{i\mathcal{V}_j})\right|\right\|_{L_p(\mathcal{D})}$$

is a partion for $(-\infty,\infty)$, thus

$$\left\|\mathcal{K}_{\mathcal{G}}(\mathfrak{f})(z)-\mathcal{K}_{\mathcal{G}}(\mathfrak{f})(z_0)\right\|_{L_p(\mathcal{D})}$$

$$\le c(p)e^{-|\mathcal{V}_j|/\mathcal{G}}\left|\mathcal{V}_j-\mathcal{V}_{j+1}\right|\frac{1}{n}\sum_{j=1}^{\infty}\frac{1}{2\mathcal{G}}\omega_1(\mathfrak{f};|z-z_0|)_{\overline{\mathcal{D}}}$$

$$\le c(p)\frac{1}{2\mathcal{G}}\int_{-\infty}^{+\infty}\omega_1(\mathfrak{f};|z-z_0|)_{\overline{\mathcal{D}}}\frac{1}{n}e^{-|\mathcal{V}|/\mathcal{G}}d\mathcal{V} \quad (2.2)$$

$$= c(p)\frac{1}{n}\omega_1(\mathfrak{f};|z-z_0|)_{\overline{\mathcal{D}}}.$$

This completes the proof.

(ii) Let $z \in \overline{\mathcal{D}}$ & $|h| < \beta$, we get

$$\left\|\mathcal{K}_{\mathcal{G}}(\mathfrak{f})(z)-\mathcal{K}_{\mathcal{G}}(\mathfrak{f})(z+h)\right\|_{L_p(\mathcal{D})}$$

$$\leq \left\| \frac{1}{2\mathcal{G}} \int_{-\infty}^{+\infty} \left| \mathfrak{f}(ze^{i\mathcal{V}}) - \mathfrak{f}\left((z+h)e^{i\mathcal{V}}\right) \right| e^{-|\mathcal{V}|/\mathcal{G}} d\mathcal{V} \right\|_{L_p(\mathcal{D})}$$

$$\leq \left\| \frac{1}{2\mathcal{G}} \int_{-\infty}^{+\infty} \left| \Delta_h^1 \mathfrak{f}(ze^{i\mathcal{V}}) \right| e^{-|\mathcal{V}|/\mathcal{G}} d\mathcal{V} \right\|_{L_p(\mathcal{D})}$$

$$\leq \left\| \frac{1}{2\mathcal{G}} \sum_{j=1}^{\infty} \left| \Delta_h^1 \mathfrak{f}(ze^{i\mathcal{V}_j}) \frac{1}{n} \right| e^{-|\mathcal{V}_j|/\mathcal{G}} |\mathcal{V}_j - \mathcal{V}_{j+1}| \right\|_{L_p(\mathcal{D})}$$

where $\mathcal{V}_j$ is a partion for$(-\infty, \infty)$.

$$\left\| \mathcal{K}_{\mathcal{G}}(\mathfrak{f})(z) - \mathcal{K}_{\mathcal{G}}(\mathfrak{f})(z+h) \right\|_{L_p(\mathcal{D})}$$

$$\leq c(p) \sum_{j=1}^{\infty} \left\| \frac{1}{2\mathcal{G}} \left| \Delta_h^1 \mathfrak{f}(ze^{i\mathcal{V}_j}) \frac{1}{n} \right| e^{-|\mathcal{V}_j|/\mathcal{G}} |\mathcal{V}_j - \mathcal{V}_{j+1}| \right\|_{L_p(\mathcal{D})}$$

$$\leq c(p) \sum_{j=1}^{\infty} \frac{1}{2\mathcal{G}} \omega_1(\mathfrak{f};\beta)_{\overline{\mathcal{D}}} \frac{1}{n} e^{-|\mathcal{V}_j|/\mathcal{G}} |\mathcal{V}_j - \mathcal{V}_{j+1}|$$

$$\leq c(p) \frac{1}{2\mathcal{G}} \int_{-\infty}^{+\infty} \omega_1(\mathfrak{f};\beta)_{\overline{\mathcal{D}}} \frac{1}{n} e^{-|\mathcal{V}|/\mathcal{G}} d\mathcal{V} \tag{2.3}$$

$$= c(p) \frac{1}{n} \omega_1(\mathfrak{f};\beta)_{\overline{\mathcal{D}}}.$$

(iii) We have

$$\mathcal{K}_{\mathcal{G}}(\mathfrak{f})(z) - \mathfrak{f}(z) = \frac{1}{2\mathcal{G}} \int_{-\infty}^{+\infty} \left[\mathfrak{f}(z_1 e^{i\mathcal{V}}) - \mathfrak{f}(z)\right] e^{-\mathcal{V}/\mathcal{G}} d\mathcal{V} \tag{2.4}$$

$$= \frac{1}{2\mathcal{G}} \int_0^a \left[\mathfrak{f}(z_1 e^{i\mathcal{V}}) - \mathfrak{f}(z)\right] e^{-\mathcal{V}/\mathcal{G}} d\mathcal{V} + \frac{1}{2\mathcal{G}} \int_a^{\infty} \left[\mathfrak{f}(z_1 e^{i\mathcal{V}}) - \mathfrak{f}(z)\right] e^{-\mathcal{V}/\mathcal{G}} d\mathcal{V}$$

$$\leq 2 \frac{1}{2\mathcal{G}} \int_a^{\infty} \left[2\mathfrak{f}(ze^{i\mathcal{V}}) - 4\mathfrak{f}(z) + 2\mathfrak{f}(ze^{-i\mathcal{V}})\right] e^{-\mathcal{V}/\mathcal{G}} d\mathcal{V}$$

$$= \frac{1}{\mathcal{G}} \int_a^{\infty} \left[2\mathfrak{f}(ze^{i\mathcal{V}}) - 4\mathfrak{f}(z) + 2\mathfrak{f}(ze^{-i\mathcal{V}})\right] e^{-\mathcal{V}/\mathcal{G}} d\mathcal{V}, \ a > 0,$$

which implies

$$\left\|\mathcal{K}_{\mathcal{G}}(\mathfrak{f})(z)-\mathfrak{f}(z)\right\|_{\mathrm{L_p}(\mathcal{D})}=\left\|\frac{1}{\mathcal{G}}\int_a^{\infty}\left[2\mathfrak{f}\left(ze^{i\mathcal{V}}\right)-4\mathfrak{f}(z)+2\mathfrak{f}\left(ze^{-i\mathcal{V}}\right)\right]e^{-\mathcal{V}/\mathcal{G}}d\mathcal{V}\right\|_{\mathrm{L_p}(\mathcal{D})}$$

$$\leq\left\|\frac{1}{\mathcal{G}}\int_a^{\infty}\left|2\mathfrak{f}\left(ze^{i\mathcal{V}}\right)-4\mathfrak{f}(z)+2\mathfrak{f}\left(ze^{-i\mathcal{V}}\right)\right|e^{-\mathcal{V}/\mathcal{G}}d\mathcal{V}\right\|_{\mathrm{L_p}(\mathcal{D})}$$

$$\leq\left\|\frac{1}{\mathcal{G}}\int_a^{\infty}\left|\Delta_h^3\mathfrak{f}\left(ze^{i\mathcal{V}}\right)\right|e^{-\mathcal{V}/\mathcal{G}}d\mathcal{V}\right\|_{\mathrm{L_p}(\mathcal{D})}$$

$$\leq\left\|\frac{1}{\mathcal{G}}\sum_{j=1}^{\infty}\left|\Delta_h^3\mathfrak{f}\left(ze^{i\mathcal{V}_j}\right)\frac{1}{n}\right|e^{-|\mathcal{V}_j|/\mathcal{G}}\left|\mathcal{V}_j-\mathcal{V}_{j+1}\right|\right\|_{\mathrm{L_p}(\mathcal{D})}$$

where $\mathcal{V}_j$ is a partion for$(-\infty,\infty)$.

$$\left\|\mathcal{K}_{\mathcal{G}}(\mathfrak{f})(z)-\mathfrak{f}(z)\right\|_{\mathrm{L_p}(\mathcal{D})}$$

$$\leq c(p)\sum_{j=1}^{\infty}\left\|\frac{1}{\mathcal{G}}\left|\Delta_h^3\mathfrak{f}\left(ze^{i\mathcal{V}_j}\right)\frac{1}{n}\right|e^{-|\mathcal{V}_j|/\mathcal{G}}\left|\mathcal{V}_j-\mathcal{V}_{j+1}\right|\right\|_{\mathrm{L_p}(\mathcal{D})}$$

$$\leq c(p)\sum_{j=1}^{\infty}\frac{1}{\mathcal{G}}\omega_3(\mathfrak{f};\mathcal{V})_{\mathcal{D}}\frac{1}{n}e^{-|\mathcal{V}_j|/\mathcal{G}}\left|\mathcal{V}_j-\mathcal{V}_{j+1}\right|$$

$$=c(p)\frac{1}{\mathcal{G}}\int_a^{+\infty}\omega_3\left(\mathfrak{f};\frac{\mathcal{V}}{\mathcal{G}}\cdot\mathcal{G}\right)_{\mathcal{D}}\frac{1}{n}e^{-\mathcal{V}/\mathcal{G}}d\mathcal{V} \tag{2.5}$$

$$\leq c(p)\left(\frac{1}{\mathcal{G}}\int_a^{+\infty}\left[1+\frac{\mathcal{V}}{\mathcal{G}}\right]^2\frac{1}{n}e^{-\mathcal{V}/\mathcal{G}}d\mathcal{V}\right)\omega_3(\mathfrak{f};\mathcal{G})_{\mathcal{D}}\leq c(p)\psi\omega_3(\mathfrak{f};\mathcal{G})_{\mathcal{D}}$$

where $\psi=\frac{1}{\mathcal{G}}\int_a^{+\infty}\left[1+\frac{\mathcal{V}}{\mathcal{G}}\right]^2\frac{1}{n}e^{-\mathcal{V}/\mathcal{G}}d\mathcal{V}$ ■

3. Conclusion

In this work, the definition of $\mathrm{L_p}(\mathcal{D})$ was presented to study the properties of the complex Picard approximation. We used continuous complex effects and linked them to the approximation. We obtained some approximate properties of the complex singular integrals $\mathcal{K}_{\mathcal{G}}(\mathfrak{f})(z)$.

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