

Complex Dynamics of Relaxed Newton's Method and its Application to Fourth-Degree Polynomials

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Abstract

The Newton-Raphsen method is an important numerical method for finding the roots of nonlinear equations and is considered one of the best numerical methods in terms of convergence. Furthermore, the relaxed Newton method, used to find the roots of nonlinear equations with recurring roots, is of great importance. Therefore, we used this method in this research to study its complex dynamics. We also demonstrated the coupling of the relaxed Newton method, which uses a single fourth root to obtain a fourth root, via a linear fractional transformation using a Riemann sphere, with recurrence rates of the quadratic equation $q(a) = a^4 - 0.75$. The fractional transformation, the Möbius transformation, was also applied to complex quadratic polynomials with recurring roots.

Keywords: Relaxed Newton's method, Möbius transformation, fourth-degree polynomials, Multiple roots.

الديناميكيات المعقدة لطريقة نيوتن المرخاة وتطبيقاتها على متعددات الحدود من الدرجة الرابعة

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الخلاصة

تُعدّ طريقة نيوتن رافسن من الطرق العددية المهمة لإيجاد جذور المعادلات غير الخطية، وتُعتبر من أفضل الطرق العددية من حيث التقارب. علاوةً على ذلك، تُعدّ طريقة نيوتن المرخاة، المُستخدمة لإيجاد جذور المعادلات غير الخطية ذات الجذور المُتكررة، ذات أهميةٍ بالغة. لذلك، استخدمنا هذه الطريقة في هذا البحث لدراسة ديناميكياتها المُعقدة. كما بيّنا اقتران طريقة نيوتن المُرخفة، التي تستخدم جذراً رابعاً واحداً للحصول على جذرٍ رابع، من خلال تحويل كسري خطي باستخدام كرة ريمان، مع تكرارات المعادلة الرباعية $q(a) = a^4 - 0.75$. كما طُبّق التحويل الكسري، تحويل موبوس، على كثرات الحدود الرباعية المُعقدة ذات الجذور المُتكررة.

1. Introduction

Newton's approach involves iterating The formula used to make an approximation the complex or true root of a polynomial $g(a)$.

$$N(a) = z - \frac{h(a)}{h'(a)}$$

By making a preliminary approximation, a_0 , adequately near a root of h , The series of iterations, $a_{k+1} = N(a_k)$, will approach the root. The sequence converges to a quartic ally if the root is simple, However, The convergence is only linear when the root is an amount of one [1–4]. It is possible to enhance the convergence to numerous roots by modifying Newton's

approach. If it is known that $h(a)$ has several roots of order precisely r , after that, use Newton's approach to $\sqrt[r]{h(a)}$ to get one

$$N_r(a) = a - \frac{(h(a))^{\frac{1}{r}}}{\frac{1}{r}(h(a))^{\frac{1}{r}-1}h'(a)} = a - \frac{rh(a)}{h'(a)} \quad (1-1)$$

This is sometimes referred to as Newton's approach for an order's root r or the method of relaxed Newton. A root of order r will be reached quartically by this relaxed Newton's approach.

Every polynomial $h(a)$ on the complex Riemann sphere has a dynamical system defined by Newton's method. In [5-8], Haeseler and Peitgen utilize Newton's approach to sum up the complex dynamics of a function of rationality and talk about the basins of attractions of the roots. We investigate the areas of attraction underlying the relaxed Newton's approach. The use of the relaxing Newton's approach for a dual root to a quadratic with a dual root is fully covered, and we demonstrate that a well-known Julia's set is conjugate to the dynamics. For a polynomial of degree $r+1$, the relaxed Newton's technique yields a result comparable to this one for a root of order r .

Furthermore, we demonstrate that the Julia's sets of quartics are conjugate to the Benzinger, Burns, & Palmore sets of Julia [2]. when For the family of functions, the conventional Newton's approach is used $(a-1)(a + \mathbf{a})^a$.

2. Technique Relaxed Newton's method for obtaining a quartic root

This section contains one double root for a polynomial q of degree p . Next, we must demonstrate that a double root's immediate basin is simply linked. To achieve this, we start with the following premise.

Proposition 2.1. [5-8] Let c_i be the sole real double root of q , and allow the remaining roots to be genuine, fundamental roots. Next, the interval $[\mu_i, \mathcal{E}_i]$ is represented by itself., $N_a[\mu_i, \mathcal{E}_i][\mu_i, \mathcal{E}_i]$, where μ_i, \mathcal{E}_i have two open critical points situated within its immediate basin of c_i .

Theorem 2.2.[7] Assume $q(a) = b_p a^p + b_{p-1} a^{p-1} - 1 + \dots + b_1 a + b_n$ be a polynomial that has unique and real roots $x_k, 1 \leq k \leq p$, and Assume that N is the Newton's function related to $q(a)$. Next, the direct basin of each $x_k, 1 \leq k \leq p$, is just connected.

Theorem 2. 3. Let c_i be the sole real double root of q as stated in Proposition 2.1. Then, the direct basin of c_j is simply connected (just connected)

Proof. This proof's concept is identical to that of theorem (2.2). However, we must acknowledge that, the fixed point in this instance a_j of N is not critical point. Additionally, we must remember that c_j contains two preimages.

Theorem 2.4. The technique of relaxed Newton, N_2 a linear fractions translation on the Riemann sphere converts any cubic solution with a double root to the quartic's iterations $q(a)=a^4 -3/4$.

Proof. Allow the cubic to be $h(a)=(a - c)^4(a - d)$, where the complex numbers c and d are different. When applied to this cubic, Newton's double root approach is

$$N_2(a) = \frac{4h(a)}{h'(a)} = \frac{(4d-3a)c-a^2}{-5a+c+4d}$$

as well as its derivative is

$$N_2' = - \frac{(3c - 8d + 5a)(c - a)}{(-5a + c + 4d)^2}$$

The stationary points of (N_2) are $[a = c]$ & $[z = d]$. Since $N_2'(c) = 0$, c is a really attractive static point and, since $N_2'(d) = -3$, d is a neutral point that is stable. The crucial aspects of N_2 are $a = c$ & $z = \frac{(4d - c)}{3}$. The quartic's crucial points $q(a) = a^2 + m$ the Riemann sphere contains $[a = 0]$ and $[a = \infty]$, while $z = \infty$ is a highly appealing stationary point as well. The conversion of fraction linear, often known as the Möbius transformation.

$$g(a) = \frac{3a + c - 4d}{2(a - c)}$$

sends m to ∞ and $\frac{(4d - c)}{3}$ to 0 , and therefore illustrates the key elements of N_2 . In contrast, the quartic p . Reverse the map. N_2 , to use the quartic through its transformation h .

$$q(a) = g \circ N_2 \circ g^{-1}(a) = a^4 - \frac{3}{4}$$

The use of the constant factor in g allowed the ratio to of a^2 became 1.

3. Applications

Newton's approach's relaxed dynamics N_2 is the dynamics underneath the Riemann surface map h that is conjugate to q .

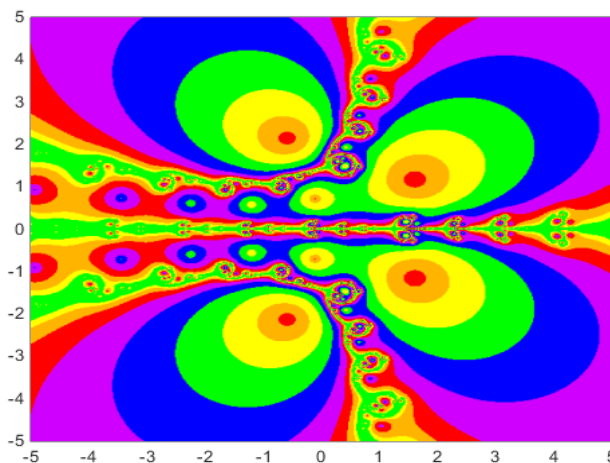


Figure -1 The attraction basin is shown by the white region after 100 iterations, of the root of quartic -1 , when N_2 is used for $(a + 1)^4 a$.

Figure 1 depicts the quartic root's attraction basin in The intricate plane. When using Newton's approach to relax N_2 is used on the cubic $h(a) = a(a + 1)^4$. Every starting value in the white area will approach the quartic root $c = -1$, whereas the black region's starting values do not converge to inside 0.001 of a root in 100 iterations. The quartic conjugate $q(a) = a^4 - \frac{3}{4}$

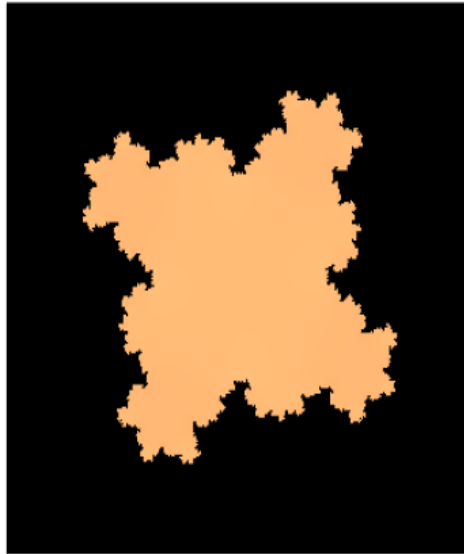


Figure -2 Julia's set with $p(z) = z^4 - 0.75$.

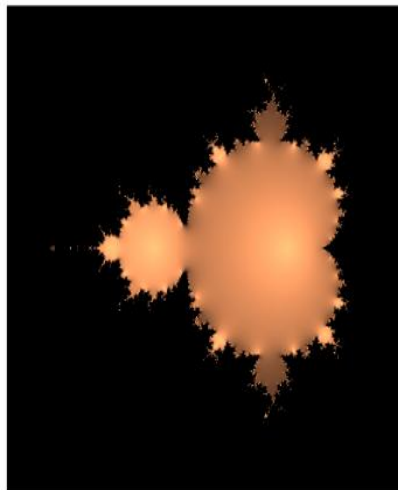


Figure -3 Identical to Figure 1, with the exception of 1195 points

The easy root 0 is iterated 0.01 times before reaching the gray region. results in the Julia's set that is filled in and displayed in Figure 2. The edge of the dark area is actually the Julia set. The dark area in Figure1 is far left is $(c+2d)/3=1/3$ and under h, this is mapped, to $-3/2$, It is Julia's set highest point in Figure 2. The Julia's set right-hand side, $3/2$, originates at the infinity point in (Figure 1).

In the Mandelbrot set, the quartic q is equivalent to point $3/4$ at the intersection of the largest circle and the cardioid. This quartic's dynamics are widely recognized. At the Julia set's extreme right, there is a repellent stationary point at $3/2$ and a neutral stationary point at $1/2$. That image represents the fundamental root b . The map $q(a) = \frac{3}{4}z^2$ is also the map's conjugate $a \rightarrow -a + a^4$, as seen in [3] with the origin's neutral stationary point. The flower's Theorem [3,4] states that there are petals around the neutral fixed point inside the black area. The flower's Theorem [3,4] states that there are petals surrounding a neutral, stationary point within the black region.

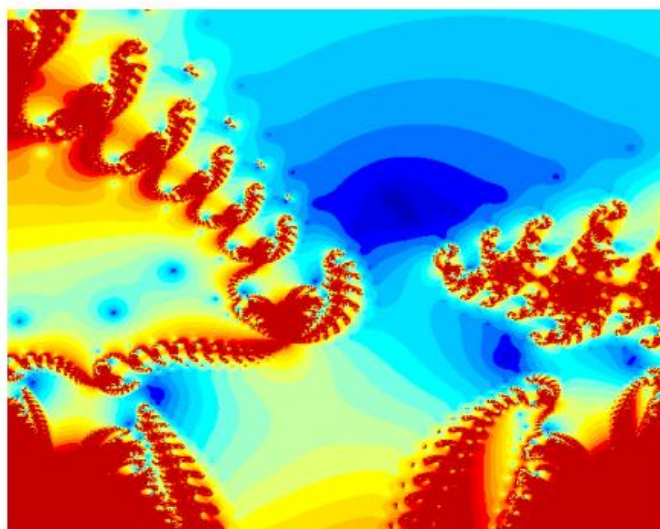


Figure -4 For $(a + 1)^2 (a^4 + 0.25)$

The basins of attraction for N_2 were applicable. Following a thousand iterations, the points with shades of white, light gray, and dark gray approach the roots of roots $-1, \frac{1}{2} - \frac{1}{2}i, \frac{1}{2} + \frac{1}{2}i, -\frac{1}{2} - \frac{1}{2}i$ and $-\frac{1}{2} + \frac{1}{2}i$, respectively, by less than 0.01.

point, which approaches that fixed point. On the other hand, a series of repelling periodic points in the Julia set converge to the neutral fixed point.

As a result, As seen in Figure 3, In fact, the points in Figure 1 is blue region converge to the root $d=0$, albeit very slowly. However, There isn't a whole neighborhood of d where every point converges to d , because the Julia set crosses any such neighborhood. There will be no convergence of points that begin on the Julia set, yet will remain perilous across iterations of N_2 .

Newton's method for families of functions is discussed by Palmore, Burns, and Benzinger [2][9].

$$(f_a(a) = (a + a)^a (a-1) \tag{2-2}$$

For $a = 1/2$, The function is supplied in this manner $(a - 1)\sqrt{(a + 0.5)}$ and the Newtonian attraction basins are depicted in [2] [10]. Since equation (1) demonstrates that the flexible Newton's technique, N_2 , for $(a - 1)^4 (a + 0.5)$ is similar to Newton's approach, N , for $(a - 1)\sqrt{(a + 0.5)}$.

4. Root of order r

Apply N_r to $h(a)=g(a)(a-c)^k$ to find out more broadly what occurs to the relaxed Newton's technique. Assume that g has limited derivatives and that c is not a root of g [7-11]. The order of the root c is k .

$$N_r'(c) = 1 - \frac{r}{k}$$

In order for r to be a super attractive fixed point, $k=r$ must be met. Otherwise, if c is an unpleasant fixed point. and if $k > \frac{r}{2}$, $|N_r'(c)| < 1$ As the cubic above illustrates, if $k = \frac{r}{2}$, $|N_r'(c)|=1$ hence the fixed point c is neutral. Despite the fact that c is a desirable fixed point for N_r and $k < \frac{r}{2}$, $|N_r'(c)| > 1$, the convergence is only linear and not quartic.

When we use the relaxed Newton's approach, Every single simple root will be a neutral fixed point, N_2 , to a function that has only one dual root and no further multiple roots. For instance, Figure 4 shows the regions of attractions for a quartic with a single dual root. The relaxed Newton's method is a simple extension of Theorem (2.1) and its proof N_r . When Theorem (2.4) is used with polynomials of any degree. $(r+1)$ with an order r root, the relaxed Newton's technique, N_r , is, equivalent to the quartic repetitions by the linear fractional translation on a Riemann sphere.

$$p(a) = a^4 + \frac{1-r^2}{4}$$

Specifically, Julia's set t is the line segment around -2 and 2 on the real axis. of $a^4 - 2$ when $r=3$. With the exception of a straight cut from $(c+3d)/4$ through d to ∞ , the entire complex plane is the triple root's attraction basin of $(a-c)^3 (a-d)$ under N_3 . As r values increase, the

The Julia set is not related at all. Theorem (2.4) can be rewritten as follows assuming α to be a positive real integer because when applied to $h(z)$, the standard Newton's approach is identical to the relaxed Newton's method for $\sqrt[n]{h(a)}$.

Theorem (2.3): The conventional Newton's approach, N , was used to $(a-c)(a-d)^a$, is equivalent to the quartic iterations on the Riemann sphere by a linear fractional translation.

$$p(a) = a^4 + \frac{1}{4} - \frac{1}{4a^2}$$

The Julia's sets that Benzinger, Burns, and Palmore [2] got by applying Newton's approach to the functions in equation (2-2) are explained by this theorem. Given that they conjugate to the quartic's Julia's set $q(a)=a^4 + c$ for $-2 < c \leq 1/4$ Julia's sets of f_a , for $1/3 < a \leq \infty$, are related.

5- Conclusion

The most significant finding in the realm of complex dynamics was that the Julia group is continuous and conjugate to the Julia's set of the quartic. This was achieved by using Newton's handy technique to fourth-degree polynomials. $q(z) = z^4 + c$ for $-2 < c \leq 1/4$ the Julia sets of f_a , for $1/3 < a \leq \infty$, are related.

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