

A New Family of Extended Rayleigh Distribution: Properties and Estimation

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Abstract

This paper presents a new type of continuous family distribution constructed on the truncated Rayleigh-Rayleigh distribution $[0,1]$. This distribution addresses the challenges posed by increasing data complexity, particularly in failure modeling, where it provides a suitable alternative when the normal distribution is inadequate. Using the maximum likelihood method (MLE) for parameter estimation, simulations were conducted to experimentally determine the behavior of the TR-RD parameters with different sample sizes ($n = 10, 20, 50, 100$) and $(200, 300, 400, 500)$. The simulation results demonstrated that using small parameter values with large sample sizes yields a clear advantage, achieving the lowest mean squared error (MSE). This paper also defines several properties of this distribution, including, for example, the r -order moment function, the reliability function, the failure rate function, the dielectric strength function, and the Shannon entropy function.

Keywords: Rayleigh distribution, truncated function, the function of r th moment, reliability, hazard rate function, Shannon entropy function.

عائلة جديدة من توزيعات رايلي الموسعة: الخصائص والتقدير

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قسم تقنيات المحاسبة، الكلية التقنية ذي قار، الجامعة التقنية الجنوبية، ذي قار، العراق

الخلاصة

قدم في هذه الورقة نوعاً جديداً من التوزيعات العائلية المستمرة المنشأة على توزيع رايلي-رايلي المقطوع $[0,1]$. وذلك لغرض معالجة التطور الحاصل بتعدد البيانات وخصوصاً في نمذجة الأعطال، حيث يعد بديلاً مناسباً عندما يكون التوزيع الطبيعي غير مناسباً. استخدمه طريقة الإمكان الأعظم MLE لتقدير المعلمات، أُجريت محاكاة لتحديد سلوك معلمات TR-RD تجريبياً بأحجام عينات مختلفة $(10, 20, 50, 100, n=)$ و $(200, 300, 400, 500, n=)$. وظهرت نتائج المحاكاة ان استخدام قيم صغيرة للمعلمات مع احجام عينات كبيرة يؤدي الى افضلية واضحة، حيث تحقق اقل قيم لمتوسط مربع الخطأ (MSE). تُعرّف هذه الورقة أيضاً بخصائص عديدة لهذا التوزيع، منها على سبيل المثال دالة العزم من الرتبة r ، ودالة الموثوقية، ودالة معدل الفشل، ودالة قوة العزل الكهربائي، ودالة إنتروبيا شانون.

1. Introduction

The importance of Raleigh distribution (RD) appeared in various fields of science and technology application, especially in the field of acoustics. Lord Raleigh was the first to introduce this distribution in (1880) by Rayleigh [15], "Rayleigh distribution has good relations with the other well-known distributions, chi-square distribution, Weibull distribution and extreme value distribution, by Sanku and Tanujit [9]. Morad and Mahdi

[4], introduced a new family of probability continuous distributions named the transmuted Weibull-G family". "They presented some valuable characterizations based on the ratio of two truncated moments and hazard function". Mustafa [12], presented a new extended from Weibull distribution with three parameters. the researcher Mohamed [11], presented a new class of distribution named Frechet- Extended distribution and they presented some of its properties. Abdulhakim [1], in this paper, the researcher proposed a new extension of the Rayleigh distribution with two parameters, called the type I half-logistic Rayleigh distribution. Chukwandi and Prince [13], The researchers presented an innovative probability distribution called the Lambert-Rayleigh (LR) distribution, with the aim of overcoming the limitations observed in traditional distributions, particularly the Rayleigh distribution. Najm and Emad [14], They introduced a new distribution called "SGR" (Semi-Circular Generalized Two-Parameter Rayleigh Distribution), a recent development based on Inverse Stereoscopic Projection (ISP). Anabike and Ebubesus [6], They developed the New Extended Rayleigh (NER) distribution by deriving it from the family of New Extended X distributions. Dawlah [5], proposed a new distribution called the Exponential Generalized Whipple-Rayleigh Distribution (EGWR), which is adaptable to different types of medical data. Alanazi and Khudhayr [2], the authors presented an innovative asymmetric formulation of the Rayleigh distribution, known as the generalized Komaraswamy-Rayleigh model.

In this paper we introduce a new probability distribution known as a [0,1] Truncated Rayleigh- Rayleigh distribution ([0,1]TR-RD).

Suppose that X follows a Rayleigh distribution, then the probability density function (PDF) and cumulative distribution function (CDF) of this distribution can be written in the following form Sanku [8] and Catherine [10];

$$f(x; \lambda) = \frac{x}{\lambda^2} e^{-x^2/(2\lambda^2)} \quad x \geq 0, \quad \lambda > 0 \quad (1)$$

The corresponding CDF, is given by;

$$F(x; \lambda) = 1 - e^{-x^2/(2\lambda^2)} \quad (2)$$

2. [0, 1] Truncated G- L Distributions

Suppose that $L(x)$ and $l(x)$ be any PDF and CDF continuous type random variable X . Suppose that $G(\cdot)$ and $g(\cdot)$ represents, the CDF and PDF respectively of any continuous random variable bounded by $[0, \infty)$.

The general form of CDF for a new class depends on substitute G with L given as Mohamed [7] and Salah [3]

$$F(x)_{TG-L} = \frac{G[L(x)]-G[0]}{G[1]-G[0]} \quad (3)$$

Since we have $G[0] = 0$, Then cdf in (3);

$$F(x)_{TG-L} = \frac{G[L(x)]}{G[1]} \quad (4)$$

And its associated pdf, $f(x) \frac{d}{dx} [F(x)_{TG-L}]$ will be,

$$f(x)_{TG-L} = \frac{g[L(x)]l(x)}{G[1]} \quad (5)$$

3. Truncated [0,1] Rayleigh Distribution

We present a new class of [0,1] truncated based on Rayleigh (RD).

Let $G(.)$ and $g(.)$ in (4) and (5), be the pdf and cdf of (RD) recall (1) and (2) with one parameter ($\lambda > 0$). We have $G(0)=0$ So,

$$G[L(x)]_{TR-L} = 1 - e^{-(L(x))^2/(2\lambda^2)} \quad , \quad G[1] = 1 - e^{-1/(2\lambda^2)}$$

$$\text{And } g[L(x)]_{TR-L} = \frac{L(x)}{\lambda^2} e^{-(L(x))^2/(2\lambda^2)}$$

Then giving to (4) , (5). Assume probability density function (PDF) and cumulative distribution function (CDF) for a new class of distribution named [0,1] Rayleigh Truncated [0,1] TR-LD) will be,

$$F(x)_{TR-L} = 1 - e^{-(L(x))^2/(2\lambda^2)} / 1 - e^{-1/(2\lambda^2)} \quad , \quad \lambda > 0, x > 0 \quad (6)$$

and,

$$f(x)_{TR-L} = \frac{L(x)}{\lambda^2} e^{-\left(\frac{L(x)^2}{2\lambda^2}\right)} / 1 - e^{-1/(2\lambda^2)} \quad , \quad \lambda > 0 \quad x > 0 \quad (7)$$

4. [0,1] Truncated Rayleigh Distribution - Rayleigh Distribution

Assume that $L(x)$ and $l(x)$, the Rayleigh Truncated Distribution by one parameter β , cdf and pdf by way of,

$$L(x; \beta) = 1 - e^{-x^2/(2\beta^2)} \quad (8)$$

$$l(x; \beta) = \frac{x}{\beta^2} e^{-x^2/(2\beta^2)} \quad (9)$$

According to (6), the CDF of new distribution called truncated [0,1] Rayleigh - Rayleigh distribution ([0,1] TR-RD) given as,

$$F(x)_{TR-R} = \frac{1}{1 - e^{-1/(2\lambda^2)}} \left[1 - e^{-(1 - e^{-x^2/(2\beta^2)})^2 / 2\lambda^2} \right] \quad (10)$$

The pdf [0,1] TR-R distribution can be obtained, according to (7) as,

$$f(x)_{TR-R} = \frac{1}{\lambda^2(1 - e^{-1/(2\lambda^2)})} \left[\left(1 - e^{-x^2/(2\beta^2)} \right) e^{-\left(\frac{(1 - e^{-x^2/(2\beta^2)})^2}{2\lambda^2}\right) \frac{x}{\beta^2} e^{-x^2/(2\beta^2)}} \right] \quad (11)$$

The reliability function [0,1]TR-R distribution can be obtained as,

$$R(x)_{TR-R} = 1 - F(x)_{TR-R}$$

$$R(x)_{TR-R} = 1 - \left\{ \frac{1}{1 - e^{-1/2\lambda^2}} \left[1 - e^{-(1 - e^{-x^2/(2\beta^2)})^2 / 2\lambda^2} \right] \right\} \quad (12)$$

The hazard rate function of [0,1]TR-R distribution as,

$$H(x)_{TR-R} = \frac{f(x)_{TR-R}}{R(x)_{TR-R}}$$

$$H(x)_{TR-R} = \frac{\left\{ \frac{1}{\lambda^2(1-e^{-1/(2\lambda^2)})} \left[(1-e^{-x^2/(2\beta^2)}) e^{-\left(\frac{(1-e^{-x^2/(2\beta^2)})^2}{2\lambda^2}\right)\frac{x}{\beta^2}} e^{-x^2/(2\beta^2)} \right] \right\}}{1 - \left\{ \frac{1}{1-e^{-1/2\lambda^2}} \left[1 - e^{-(1-e^{-x^2/(2\beta^2)})^2/2\lambda^2} \right] \right\}} \quad (13)$$

5. Properties of the [0,1] TR-R D

First property that is $f(x)_{TR-R}$ is pdf,

$$\lim_{x \rightarrow 0} x \rightarrow 0$$

$$\lim_{x \rightarrow 0} F(x)_{TR-R} = \lim_{x \rightarrow \infty} \left\{ \frac{1}{1 - e^{-1/2\lambda^2}} \left[1 - e^{-(1-e^\infty)^2/2\lambda^2} \right] \right\}$$

$$= \frac{1 - e^{-1/2\lambda^2}}{1 - e^{-1/2\lambda^2}} = 1$$

5.1 r-th Moment

The function of r^{th} moment, [0,1]TR-R D, is

$\int_0^\infty x^r f(x)_{TR-R} dx$. Giving to (11), r^{th} moment of [0,1] TR-RD is given by,

$$E(X^r)_{TR-R} = \int_0^\infty x^r \left\{ \frac{\frac{1}{\lambda^2 \left(1 - e^{-\frac{1}{2\lambda^2}}\right)}}{\left[(1 - e^{-x^2/(2\beta^2)}) e^{-\left(\frac{(1-e^{-x^2/(2\beta^2)})^2}{2\lambda^2}\right)\frac{x}{\beta^2}} e^{-x^2/(2\beta^2)} \right]} \right\} dx \quad (14)$$

Let, $I = (1 - e^{-x^2/(2\beta^2)})$, $II = e^{-\left(\frac{(1-e^{-x^2/(2\beta^2)})^2}{2\lambda^2}\right)\frac{x}{\beta^2}}$

$$I = (1 - e^{-x^2/(2\beta^2)})$$

By using the formula by Mohamed [7],

$$(1 - u)^b = \sum_{i=0}^\infty (-1)^i C_i^b u^i; \quad |u| < 1, b > 0, \quad (15)$$

And $e^{-u} = \sum_{k=0}^\infty \frac{(-1)^k}{k!} u^k, \quad (16)$

According to (15) I will be,

$$I = \sum_{i=0}^{\infty} (-1)^i C_i^1 e^{\frac{-ix^2}{2\beta^2}}$$

Now,

$$II = e^{-\left(\frac{(1-e^{-x^2/(2\beta^2)})^2}{2\lambda^2}\right)}$$

According to (16) II will be,

$$II = \sum_{k=0}^{\infty} \frac{(-1)^k (1 - e^{-x^2/(2\beta^2)})^{2k}}{k! (2\lambda^2)^k}$$

Also according to (15) II will be,

$$II = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! (2\lambda^2)^k} \sum_{t=0}^{\infty} (-1)^t C_t^{2k} e^{-tx^2/2\beta^2}$$

Substitutes I and II in (14) we get,

$$E(X^r)_{TR-R} = \int_0^{\infty} x^r \left\{ \frac{1}{\lambda^2 \left(1 - e^{-\frac{1}{2\lambda^2}}\right)} \sum_{i=0}^{\infty} (-1)^i C_i^1 \right. \\ \left. \sum_{k=0}^{\infty} \frac{(-1)^k}{k! (2\lambda^2)^k} \sum_{t=0}^{\infty} (-1)^t C_t^{2k} \frac{x}{\beta^2} e^{-(i+t+1)x^2/(2\beta^2)} \right\} dx$$

$$E(X^r)_{TR-R} = \left\{ \frac{1}{\lambda^2 \left(1 - e^{-\frac{1}{2\lambda^2}}\right)} \sum_{i=0}^{\infty} (-1)^i C_i^1 \right. \\ \left. \sum_{k=0}^{\infty} \frac{(-1)^k}{k! (2\lambda^2)^k} \sum_{t=0}^{\infty} (-1)^t C_t^{2k} \frac{1}{\beta^2} \int_0^{\infty} x^{r+1} e^{-(i+t+1)x^2/2\beta^2} \right\}$$

By using the formula, $\int_0^{\infty} x^{\alpha-1} e^{-\beta x} dx = \frac{\Gamma(\alpha)}{\beta^\alpha}$

The function of r^{th} moment of TR-R distribution, is given by,

$$E(X^r)_{TR-R} = \left\{ \frac{1}{\lambda^2 \left(1 - e^{-\frac{1}{2\lambda^2}}\right)} \sum_{i=0}^{\infty} (-1)^i C_i^1 \right. \\ \left. \sum_{k=0}^{\infty} \frac{(-1)^k}{k! (2\lambda^2)^k} \sum_{t=0}^{\infty} (-1)^t C_t^{2k} \frac{(2\beta^2)^{r+2}}{\beta^2} \frac{\Gamma(r+2)}{(i+t+1)^{r+2}} \right\} \tag{17}$$

Depending on the particular $E(x^r)_{TR-R}$; ($r = 1,2,3,4$), another properties of this distribution such as, (*mean* $\mu = E(x)$), (*variance* ($var(x) = \sigma^2 = E(x^2) - (E(x))^2$)) the skewness coefficient

$$\left(SK = \frac{E[X-\mu]^3}{\sigma^3} = \frac{E(x^2)-3\mu E(x^2)+2\mu^2}{(\sigma^2)^{\frac{3}{2}}} \right) \cdot \text{Coefficient of kurtosis} \left(Kr = \frac{E(X-\mu)^4}{\sigma^4} = \frac{E(x^4)-4\mu E(x^3)+6\mu^2 E(x^2)-3\mu^3}{\sigma^4} \right)$$

can be obtained, where,

$$E(X^r)_{TR-R} = \left\{ \begin{array}{l} \frac{1}{\lambda^2 \left(1 - e^{-\frac{1}{2\lambda^2}}\right)} \sum_{i=0}^{\infty} (-1)^i C_i^1 \\ \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(2\lambda^2)^k} \sum_{t=0}^{\infty} (-1)^t C_t^{2k} \frac{(2\beta^2)^{r+2}}{\beta^2} \frac{r(r+2)}{(i+t+1)^{r+2}} \end{array} \right\}, r=1,2,3,4$$

5.2 Function of characteristic $\phi_X(t)_{TR-R}$

The characteristic function of TR-RD is given by follows,

$$E(e^{itx}) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} E(X^r)_{TR-R}$$

Therefore, the characteristic function of TR-R D is given by follows,

$$\phi_X(t)_{TR-R} = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \left\{ \begin{array}{l} \frac{1}{\lambda^2 \left(1 - e^{-\frac{1}{2\lambda^2}}\right)} \sum_{i=0}^{\infty} (-1)^i C_i^1 \\ \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(2\lambda^2)^k} \sum_{t=0}^{\infty} (-1)^t C_t^{2k} \frac{(2\beta^2)^{r+2}}{\beta^2} \frac{r(r+2)}{(i+t+1)^{r+2}} \end{array} \right\} \tag{18}$$

5.3 The function of Shannon Entropy

(SE) for TR-RD it will be presented by

$$- \int_0^{\infty} \ln(f(x)_{TR-R}) f(x)_{TR-R} dx$$

By taking the natural logarithm of the PDF in equation (7), the result will be obtained.

$$\ln(f(x)_{TR-R}) = \ln \left\{ \frac{\frac{L(x)}{\lambda^2} e^{-\left(\frac{L(x)}{2\lambda^2}\right) l(x)}}{1 - e^{-1/(2\lambda^2)}} \right\}$$

$$\ln f(x)_{TR-R} = \left\{ \begin{array}{l} \ln L(x) - \ln \lambda^2 - \frac{(L(x))^2}{2\lambda^2} \\ + \ln l(x) - \ln \left(1 - e^{-\frac{1}{2\lambda^2}}\right) \end{array} \right\} \tag{19}$$

Let, $V = \ln(L(x))$ and $VI = (L(x))^2$, $VII = \ln(l(x))$

$$V = \ln(L(x))$$

$$V = \ln(1 - e^{-x^2/(2\beta^2)})$$

By using the formulas [7], $\ln(1 - u) = -\sum_{i=0}^{\infty} \frac{u^i}{i}$; $|u| < 1$, $x > 0$

$$V = -\sum_{i=0}^{\infty} \frac{e^{-ix^2/2\beta^2}}{i}$$

According (16) V will be,

$$V = -\sum_{i=0}^{\infty} \frac{1}{i} \sum_{k=0}^{\infty} \frac{(-1)^k i^k}{k! (2\beta^2)^k} x^{2k}$$

Now $VI = (L(x))^2$

$$VI = (1 - e^{-x^2/(2\beta^2)})^2$$

According to (15) VI will be,

$$VI = \sum_{s=0}^{\infty} (-1)^s C_s^2 e^{-sx^2/(2\beta^2)}$$

According to (16) VI will be,

$$VI = \sum_{s=0}^{\infty} (-1)^s C_s^2 \sum_{t=0}^{\infty} \frac{(-1)^t}{t!} \frac{s^t}{(2\beta^2)^t} x^{2t}$$

Now,

$$VII = \ln(l(x))$$

$$VII = \ln\left(\frac{x}{\beta^2} e^{-\frac{x^2}{2\beta^2}}\right)$$

$$= \ln(x) - \ln(2\beta^2) - \frac{x^2}{2\beta^2}$$

By using the formula [7], $\ln(x) = 2 \sum_{n=0}^{\infty} \frac{(x-1)^{2n+1}}{(x+1)^{2n+1}}$, $x > 0$,

and $(a + u)^{-n} = \sum_{i=0}^{\infty} C_i^{-n} a^{-n-i} u^i$ we get,

$$VII = 2 \sum_{n=0}^{\infty} (x-1)^{2n+1} / (x+1)^{2n+1} - \ln(2\beta^2) - \frac{x^2}{2\beta^2}$$

By substituting V, VI, and VII into equation (19), we obtain,

The Shannon entropy function of the TR-R distribution can be expressed as follows:

$$SH_{TR-R} = - \int_0^\infty \left\{ \begin{aligned} & - \sum_{i=0}^\infty \frac{1}{i} \sum_{k=0}^\infty \frac{(-1)^k i^k}{k!(2\beta^2)^k} x^{2k} - \ln(\lambda^2) \\ & - \frac{1}{2\lambda^2} \sum_{s=0}^\infty (-1)^s C_s^2 \sum_{t=0}^\infty \frac{(-1)^t}{t!} \frac{s^t}{(2\beta^2)^t} x^{2t} \\ & + 2 \sum_{n=0}^\infty \sum_{m=0}^{2n+1} C_m^{2n+1} (-1)^{(2n+1)-m} \\ & \left(\sum_{w=0}^\infty C_w^{-(2n+1)} x^{w+m} - \ln(2\beta^2) - \frac{x^2}{2\beta^2} - \ln \left(1 - e^{\frac{-1}{2\lambda^2}} \right) \right) \end{aligned} \right\} f(x)_{TR-R} dx$$

Therefore, the function of Shannon entropy of TR-RD in (7) given by,

$$SH_{TR-R} =$$

$$\left\{ \begin{aligned} & \sum_{i=0}^\infty \frac{1}{i} \sum_{k=0}^\infty \frac{(-1)^k i^k}{k!(2\beta^2)^k} \int_0^\infty x^{2k} f(x)_{TR-R} dx + \ln(\lambda^2) \\ & + \frac{1}{2\lambda^2} \sum_{s=0}^\infty (-1)^s C_s^2 \sum_{t=0}^\infty \frac{(-1)^t}{t!} \frac{s^t}{(2\beta^2)^t} \int_0^\infty x^{2t} f(x)_{TR-R} dx \\ & - 2 \sum_{n=0}^\infty \sum_{m=0}^{2n+1} C_m^{2n+1} (-1)^{(2n+1)-m} \\ & \left(\sum_{w=0}^\infty C_w^{-(2n+1)} \int_0^\infty x^{w+m} f(x)_{TR-R} dx + \ln(2\beta^2) + \frac{\int_0^\infty x^2 f(x)_{TR-R} dx}{2\beta^2} + \ln \left(1 - e^{\frac{-1}{2\lambda^2}} \right) \right) \end{aligned} \right\}$$

$$SH_{TR-R} = \left\{ \begin{aligned} & \sum_{i=0}^\infty \frac{1}{i} \sum_{k=0}^\infty \frac{(-1)^k i^k}{k!(2\beta^2)^k} E(X^{2k}) + \ln(\lambda^2) \\ & + \frac{1}{2\lambda^2} \sum_{s=0}^\infty (-1)^s C_s^2 \sum_{t=0}^\infty \frac{(-1)^t}{t!} \frac{s^t}{(2\beta^2)^t} E(X^{2t}) \\ & - 2 \sum_{n=0}^\infty \sum_{m=0}^{2n+1} C_m^{2n+1} (-1)^{(2n+1)-m} \\ & \left(\sum_{w=0}^\infty C_w^{-(2n+1)} E(X^{w+m}) + \ln(2\beta^2) + \frac{1}{2\beta^2} E(X^2) + \ln \left(1 - e^{\frac{-1}{2\lambda^2}} \right) \right) \end{aligned} \right\}$$

Where,

$$E(X^{w+m}), E(X^{2k}), E(X^2) \text{ and } E(X^{2t}) \text{ as in (17) with } (r = 2, 2t, 2k, w + m)$$

5.4 The function of Stress -Strength

Spouse X and Y represent two independent random variables (r.v.s) for Stress and Strength, respectively, defined within the interval [0,1] with a TR-RD (Two-Parameter Reliability Distribution). They are characterized by different parameters, specifically λ_1 and β_1 . The relationship or function between Stress and Strength is derived as follows:

$$SS_{TR-R} = P_r(Y < X) = \int_0^\infty f_X(x)_{TR-R} F_Y(x) dx,$$

where,

$$F_Y(x) = \left\{ \frac{1}{1 - e^{-\left(\frac{1}{2\lambda_1^2}\right)}} \left[1 - e^{-(1 - e^{-x^2/(2\beta_1^2)})^2 / 2\lambda_1^2} \right] \right\}$$

According to(16) $F_y(x)$ will be,

$$F_y(x) = \left\{ \frac{1}{1-e^{-1/2\lambda_1^2}} \sum_{i=0}^{\infty} (-1)^i C_i^1 \sum_{k=0}^{\infty} \frac{(-1)^{ki} k^k}{k!(2\lambda_1^2)^k} (1 - e^{-x^2/(2\beta_1^2)})^{2k} \right\}$$

According to (15) $F_y(x)$ will be,

$$F_y(x) = \left\{ \frac{1}{1-e^{-\frac{1}{2\lambda_1^2}}} \sum_{i=0}^{\infty} (-1)^i C_i^1 \sum_{k=0}^{\infty} \frac{(-1)^{ki} k^k}{k!(2\lambda_1^2)^k} \right. \\ \left. \sum_{t=0}^{\infty} (-1)^t C_t^{2k} e^{-tx^2/2\beta_1^2} \right\}$$

According to (16) $F_y(x)$ will be,

$$F_y(x) = \left\{ \frac{1}{1-e^{-\frac{1}{2\lambda_1^2}}} \sum_{i=0}^{\infty} (-1)^i C_i^1 \sum_{k=0}^{\infty} \frac{(-1)^{ki} k^k}{k!(2\lambda_1^2)^k} \right. \\ \left. \sum_{t=0}^{\infty} (-1)^t C_t^{2k} \sum_{s=0}^{\infty} \frac{(-1)^{st} t^s}{s!(2\beta_1^2)^2} x^{2s} \right\} \tag{20}$$

Therefore, by using (20), the stress -strength of the [0,1] TR-RD is provided by,

$$SS_{TR-R} = \left\{ \frac{1}{1 - e^{-\frac{1}{2\lambda_1^2}}} \sum_{i=0}^{\infty} (-1)^i C_i^1 \sum_{k=0}^{\infty} \frac{(-1)^{ki} k^k}{k!(2\lambda_1^2)^k} \right. \\ \left. \sum_{t=0}^{\infty} (-1)^t C_t^{2k} \sum_{s=0}^{\infty} \frac{(-1)^{st} t^s}{s!(2\beta_1^2)^2} E(X^{2s}) \right\}$$

Where,

$E(X^{2s})$ as in (17) with $(r = 2s)$.

6. Estimation parameters of the TR – R Distribution

We obtain the maximum likelihood estimate (MLE) of the TR – R parameters. Let x_1, x_2, \dots, x_n be a random sample size from the probability function $X \sim TR - R(\lambda, \beta)$, likelihood function is,

$$L(\lambda, \beta \setminus X) = \prod_{i=0}^n [f(x_i \setminus \lambda, \beta)] \\ = \prod_{i=0}^n \left(\frac{1}{\lambda^2(1-e^{-1/(2\lambda^2)})} \left[(1 - e^{-x^2/(2\beta^2)}) e^{-\left(\frac{(1-e^{-x^2/(2\beta^2)})^2}{2\lambda^2}\right) \frac{x}{\beta^2} e^{-x^2/(2\beta^2)}} \right] \right) \\ = \left(\frac{1}{\lambda^2(1-e^{-1/(2\lambda^2)})} \right)^n \prod_{i=0}^n \left((1 - e^{-x^2/(2\beta^2)}) e^{-\left(\frac{(1-e^{-x^2/(2\beta^2)})^2}{2\lambda^2}\right) \frac{x}{\beta^2} e^{-x^2/(2\beta^2)}} \right)$$

So, log-likelihood function is,

$$\ln L(\lambda, \beta \setminus X) = n \ln \left(\frac{1}{\lambda^2 (1 - e^{-1/(2\lambda^2)})} \right) \sum_{i=0}^n \ln \left((1 - e^{-x^2/(2\beta^2)}) e^{-\left(\frac{(1 - e^{-x^2/(2\beta^2)})^2}{2\lambda^2}\right) \frac{x}{\beta^2} e^{-x^2/(2\beta^2)}} \right)$$

The MLEs $\hat{\lambda}$, and $\hat{\beta}$, are obtained respectively by solving the four nonlinear equations,

$$\frac{\partial \ln L(\lambda, \beta \setminus X)}{\partial \beta} = \left\{ \begin{array}{l} n \ln \left(\frac{1}{\lambda^2 (1 - e^{-1/(2\lambda^2)})} \right) \\ \sum_{i=0}^n \left[\ln(1 - e^{-x^2/(2\beta^2)}) - \left(\frac{(1 - e^{-x^2/(2\beta^2)})^2}{2\lambda^2} \right) \frac{x}{\beta^2} e^{-x^2/(2\beta^2)} \right] \end{array} \right\} = 0$$

$$\frac{\partial \ln L(\lambda, \beta \setminus X)}{\partial \lambda} = \left\{ \begin{array}{l} n[-\ln(\lambda^2) + \ln(1 - e^{-1/(2\lambda^2)})] \\ \sum_{i=0}^n \left[x^2/(2\beta^2) - \left(\frac{(1 - e^{-x^2/(2\beta^2)})^2}{2\lambda^2} \right) \frac{x}{\beta^2} e^{-x^2/(2\beta^2)} \right] \end{array} \right\} = 0$$

7. Simulated Data

A random variable x which has TR-RD will be simulated through by numerical solve equation,

$$U(1 - e^{-1/2\lambda^2}) - \left[1 - e^{-(1 - e^{-x^2/(2\beta^2)})^2/2\lambda^2} \right] = 0$$

Where U represent ([0,1] uniform distribution).

A simulation was conducted to experimentally determine the behavior of the parameters TR-RD with different sample sizes ($n = 10, 20, 50, 100$) and ($n = 200, 300, 400, 500$). The random variable TR-RD is simulated by applying the formula in (11) using the default values for the parameters given in Table with $\lambda = 0.4, 1.4$; $\beta = 0.4, 1.4$. The trial size is $Z = 1000$. The root mean square error (*MSE*) is used as a comparison and evaluation measure, where

$$MSE = \frac{1}{1000} \sum_{Z=1}^{1000} (\hat{\zeta}_Z - \zeta)^2; \quad \zeta = \lambda \text{ or } \beta$$

From the experimental results we can see:

- In every case, the value of *MSE'* decreases as the sample size increases. This outcome aligns with statistical theory. If the default value of is low the most accurate is $n= 10, 20$ and 50 , The most accurate is always associated with the parameters λ and β respectively.
- With high default values, the *MSE* values increase for all sample sizes.
- The values of λ and β of the *MSE* parameters increase as the default value increases.
- The results show that as the parameter values increase, the *MSE* values increase for all sample sizes, with a sample size of 500 showing the best performance in all cases.
-

Table 1- Empirical *MSE* values of parameter estimates for TR-RD for different cases with default values $\lambda = 0.4, 1.4$ and $\beta = 0.4, 1.4$ and Sample size (10,20,50,100)

Default Values		Sample size	MSE values		
λ	β	n	$\hat{\lambda}_{ML}$	$\hat{\beta}_{ML}$	
0.4	0.4	10	0.0487	0.0542	
		20	0.0362	0.0428	
		50	0.0194	0.0227	
		100	0.0107	0.0168	
	1.4	0.4	10	0.0985	0.9436
			20	0.0742	0.8722
			50	0.0366	0.5211
		1.4	10	0.9763	0.0273
			20	0.7832	0.0194
			50	0.4713	0.0142
1.4	1.4	100	0.1376	0.0112	
		10	1.0972	0.8591	
		20	0.7199	0.5721	
		50	0.5821	0.4218	
		100	0.3975	0.2773	

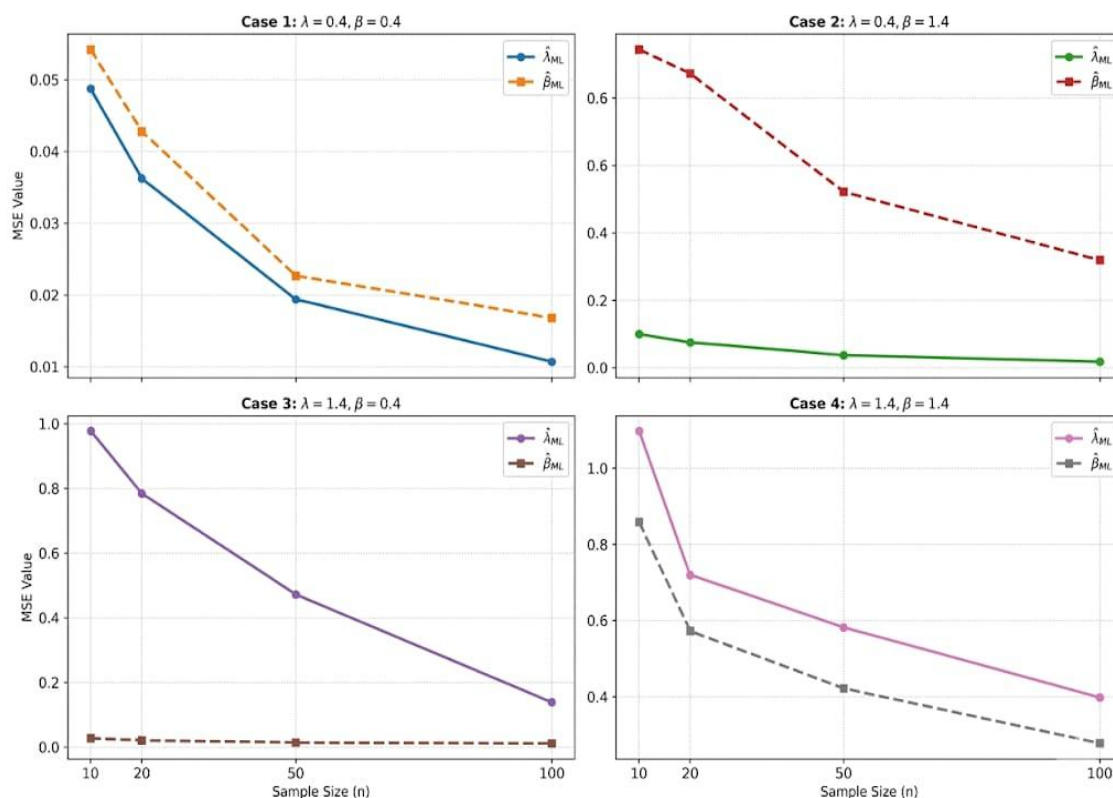


Figure --The effect of sample size (n) :The graph clearly shows the consistency of the results with the "Asymptotic Theory", where the MSE values approach zero as the sample size (n) increases.

Table 2- Empirical MSE values of parameter estimates for TR-RD for different cases with default values $\lambda = 0.4, 1.4$ and $\beta = 0.4, 1.4$ and Sample size (200,300,400,500)

Default Values		Sample size	MSE values	
λ	β	n	$\hat{\lambda}_{ML}$	$\hat{\beta}_{ML}$
0.4	0.4	200	0.0105	0.0127
		300	0.0101	0.0105
		400	0.0098	0.0068
	1.4	500	0.0083	0.0062
		200	0.0170	0.3106
		300	0.0153	0.3101
1.4	0.4	400	0.0149	0.3086
		500	0.0112	0.3036
		200	0.1321	0.0109
	1.4	300	0.1286	0.0101
		400	0.1212	0.0084
		500	0.1174	0.0051
	1.4	200	0.3288	0.2685
		300	0.3094	0.2631
		400	0.2792	0.2219
		500	0.1984	0.2072

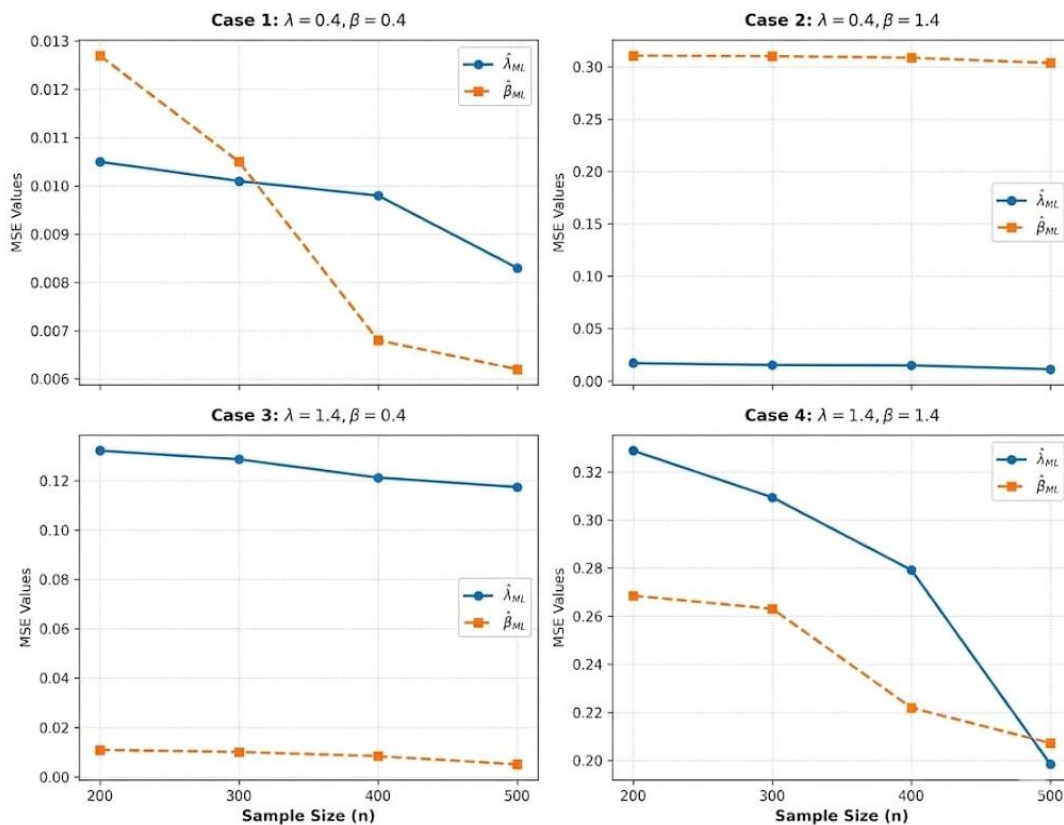


Figure -2 The effect of sample size (n): It is clear from the four curves that as we move to the right (i.e., as the sample size increases from 200 to 500), the mean squared error (MSE) values decrease for both estimators. This behaviour practically demonstrates the consistency property of maximum likelihood estimators.

7. Conclusions

In our research, we introduced a new family of continuous distributions known as the truncated [0,1] Rayleigh-Rayleigh distribution. The simulation results demonstrated that using small parameter values with large sample sizes yields a clear advantage, achieving the lowest mean squared error (MSE). We derived several key properties of the [0,1] TR-RD, including the characteristic function, rth moment, mean, variance, kurtosis, skewness, reliability function, hazard rate, and Shannon entropy function. We also deal with the determination of $SS_{TER} = P(Y < X)$ when Y and X be to two independent stress and strength respectively r vs [0,1] TR-RD with different parameters.

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