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Micro P_β −Open Set in Micro Topological Spaces

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Abstract

This paper introduces a new class of open sets within micro topological spaces, termed Micro P_β-open sets. We investigate its fundamental properties. Furthermore, we establish the exact inclusion relations, proving that any Micro P_β-open set is necessarily Micro pre-open, Micro ḃ-open, and Micro β -open. Through counterexamples, we demonstrate that these inclusions are not reversible and that the class of Micro P_β-open sets is independent of Micro α-open, Micro semi-open, Micro S_P-open and Micro S_β-open sets.

Keywords: Nano Topology, Micro Topology, Micro β – open set, Micro semi-open set, Micro P_β − open set, Micro P_β − closed set.

MSC(2020): 54A05

حول المجموعات المفتوحة المايكروية P_β في الفضاءات التبولوجية المايكروية

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الخلاصة

تقدم هذه الورقة البحثية صنفاً جديداً من المجموعات المفتوحة ضمن الفضاءات التبولوجية المايكروية، يُطلق عليها اسم المجموعات المفتوحة المايكروية من النمط P_β (Micro P_β -open sets). كما نستقصي الخصائص الأساسية لهذه المجموعات. علاوة على ذلك، نؤسس لعلاقات الاحتواء الدقيقة، حيث نُثبت أن أية مجموعة مفتوحة مايكروية P_β هي بالضرورة مجموعة مفتوحة مايكروية مسبقاً، ومفتوحة مايكروية من النمط ḃ (Micro ḃ-open) ومفتوحة مايكروية من النمط β (Micro β -open) ومن خلال الأمثلة المضادة، نوضح أن علاقات الاحتواء هذه غير عكوسة، وأن صنف المجموعات المفتوحة المايكروية P_β هو صنف مستقل عن المجموعات المفتوحة المايكروية من النمط α (Micro α-open) ، والمجموعات شبه المفتوحة المايكروية (Micro semi-open) ، والمجموعات المفتوحة المايكروية S_P (Micro S_P-open) ، والمجموعات المفتوحة المايكروية setsS_β (Micro S_β-open sets).

1. Introduction

The foundational framework for nano topological spaces was initially proposed by M. Lellis Thivagar and C. Richard (2013) [1], based on the principles of the boundary region and the upper and lower approximations, which Z. Pawlak established in 1982 [2].

Subsequently in 2019, S. Chandrasekar [3] expanded upon nano topology to formulate the framework of micro topological spaces, concurrently introducing the precise definitions for Micro semi-open alongside Micro pre-open sets.

In 2018, S. Chandrasekar and G. Swathi [4] advanced the theoretical framework by formalizing the notion of Micro α-open sets. Shortly after, H. Z. Ibrahim (2020) [5] proposed Micro β – open sets, and during that same year, T. H. Jasim and M. D. Sultan [6] proposed another related class called Micro $\dot{b}$-open sets.

Continuing this line of study, M. Maheswari et al. (2023) [7] introduced Micro S_P-open sets, which are defined based on Micro semi-open sets as a primary condition and Micro pre-closed sets as a secondary condition. More recently, in 2024, M. K. Gupta and S. Tyagi [8] proposed the concept of Micro S_β-open sets. This class also relies on Micro semi-open sets as the main condition but differs in the secondary condition, which is constructed using Micro β-closed sets. Furthermore, the study of micro topology has been extended to various new structures and applications. For instance, the concept of fibrewise micro-topological spaces has been introduced and extensively explored [9, 10], providing generalizations that have significant applications in mathematics. Recently, the notion of fibrewise micro-connected topological spaces has also been developed to investigate connectedness within these generalized structures [11].

Despite the recent development of various micro topological classes, such as Micro S_P-open and Micro S_β-open -open sets which primarily rely on Micro semi-open sets, there is a noticeable theoretical gap regarding classes constructed using Micro pre-open sets as a primary condition coupled with Micro β-closed sets. To bridge this gap, the present study introduces a novel category designated as Micro P_β-open sets. Our core scientific contribution lies in formulating this new concept, exploring its fundamental topological properties, and establishing its exact inclusion relations with previously established micro open categories. Through rigorous counterexamples, this paper provides a more comprehensive hierarchical framework for micro topological spaces.

2. PRELIMINARIES

Definition 2.1. [2] Let us assume a finite universe of discourse represented by a set W that is not empty, and let $\mathcal{R}$ denotes an equivalence relation over W. The pair $(W, \mathcal{R})$ is defined as an approximation space. All members sharing the identical equivalence class defined by $\mathcal{R}$ are considered to be indiscernible. For any subset S of W, we define the following:

1. The lower approximation of S, denoted by $L_{\mathcal{R}}(S)$, is the union of all equivalence classes that are completely contained within S Mathematically, it is defined as:

$$L_{\mathcal{R}}(S) = \bigcup_{s \in W} \{\mathcal{R}(s) : \mathcal{R}(s) \subseteq S\}$$

2. The upper approximation of S, denoted by $U_{\mathcal{R}}(S)$, is the union of all equivalence classes that have a non-empty intersection with S. Mathematically, it is defined as:

$$U_{\mathcal{R}}(S) = \bigcup_{s \in W} \{\mathcal{R}(s) : \mathcal{R}(s) \cap S \neq \emptyset\}$$

3. The boundary region of S with respect to $\mathcal{R}$, denoted by $B_{\mathcal{R}}(S)$, represents the elements that possibly, but not certainly, belong to S. It is formulated as:

$$B_{\mathcal{R}}(S) = U_{\mathcal{R}}(S) - L_{\mathcal{R}}(S)$$

Definition 2.2. [1] Consider a universe of discourse that is not empty denoted by W, and assume a relation of equivalence $\mathcal{R}$ is defined over this set. For any $S \subseteq W$, define $\mathcal{T}_{\mathcal{R}}(S) = \{W, \emptyset, L_{\mathcal{R}}(S), U_{\mathcal{R}}(S), B_{\mathcal{R}}(S)\}$, if $\mathcal{T}_{\mathcal{R}}(S)$ satisfies the axioms of topology on W, then it is named the nano topology on W with respect to S.In this case, the structured pair ($W, \mathcal{T}_{\mathcal{R}}(S)$) forms a nano topological space.

Definition 2.3. [3] Let ($W, \mathcal{T}_{\mathcal{R}}(S)$) be a nano topological space. Define $\mu_{\mathcal{R}}(S) = \{N \cup (N' \cap \mu) : N, N'$ are nano open while μ is not $\}$. Accordingly, the triplet ($W, \mathcal{T}_{\mathcal{R}}(S), \mu_{\mathcal{R}}(S)$) is known as a micro topological space, denoted by M.T.S.. Members belonging to the collection $\mu_{\mathcal{R}}(S)$ are defined as Micro open sets (denoted by M-open sets). Consequently, any set that serves as the complement to an M-open set is termed a Micro closed set, simply indicated as M-closed.

Example 2.4. Consider $U = \{1,2,3,4\}$ with $U/R = \{\{1,2\},\{3\},\{4\}\}$ and $X = \{1,4\} \subseteq U$, then $L_R(X) = \{4\}, U_R(X) = \{1,2,4\}$ and $B_R(X) = \{1,2\}$, we $\tau_R(X) = \{U, \emptyset, \{4\}, \{1,2\}, \{1,2,4\}\}$ if $\mu = \{3\}$, then $\mu_R(X) = \{U, \emptyset, \{4\}, \{3\}, \{1,2\}, \{3,4\}, \{1,2,3\}, \{1,2,4\}\}$.

Definition 2.5. [3] For any set A, its micro closure is denoted by Mcl(A) which is formulated to be the intersection of those M-closed sets that include A, that is, $\mathrm{Mcl}(A) = \bigcap \{F : A \subseteq F\}$ where F is M-closed. Similarly, the micro interior associated with A, denoted as Mint(A), is given by taking the union over every M-open subset enclosed within A, such $\mathrm{Mint}(A) = \bigcup \{G : A \supseteq G\}$ where G is M-open.

Definition 2.6. Consider an M.T.S. denoted by $(W, \mathcal{T}_{\mathcal{R}}(S), \mu_{\mathcal{R}}(S))$ with a subset $G \subseteq W$. This subset G is classified as:

1. A Micro semi-open set (denoted by Msemi-open) provided that $G \subseteq \mathrm{Mcl}(\mathrm{Mint}(G))$. Determining the complement for an Msemi-open set yields what is known as a Micro semi-closed set, (denoted by Msemi-closed) [3].

2. A Micro pre-open set (denoted by Mpre-open) whenever $G \subseteq \mathrm{Mint}(\mathrm{Mcl}(G))$. The complement corresponding to an Mpre-open set is designated as a Micro pre-closed set (denoted by Mpre-closed) [3].

3. A Micro α-open set (denoted by M α-open) provided that $G \subseteq \mathrm{Mint}(\mathrm{Mcl}(\mathrm{Mint}(G)))$. Determining the complement for an M α-open set yields what is known as a Micro α-closed set, (denoted by Mα-closed) [4].

4. A Micro β-open set (denoted by as Mβ-open) provided that $G \subseteq \mathrm{Mcl}(\mathrm{Mint}(\mathrm{Mcl}(G)))$. Determining the complement for an Mβ-open set yields what is known as a Micro β-closed set, (denoted by Mβ-closed) [5].

5. A Micro $\dot{b}$-open set (denoted by M$\dot{b}$-open) provided that $G \subseteq \mathrm{Mcl}(\mathrm{Mint}(G)) \cup \mathrm{Mint}(\mathrm{Mcl}(G))$. Taking the complement of an M$\dot{b}$-open set yields what is known as a Micro $\dot{b}$-closed set, (denoted by M $\dot{b}$-closed) [6].
6. A Micro S_p-open set (denoted by MS_P-open) provided that it constitutes M semi-open and given any $x \in G$, there is a M pre-closed set F where $x \in F \subseteq G$. Determining the complement for an MS_P-open set yields what is known as a Micro S_p-closed set (denoted by MS_p-closed) [7].
7. A Micro S_β-open set (denoted by MS_β-open) provided that it constitutes M semi-open and given any $x \in G$, there is a M β - closed set F where $x \in F \subseteq G$. Determining the complement for an MS_β-open set yields what is known as a Micro S_β-closed set (denoted by MS_β-closed) [8].

The set comprising every M pre-open (resp. M semi-open, M α-open, M β-open, M $\dot{b}$-open, M S_P-open and MS_β-open) collection is represented as $mPO(W,S)$ (resp. $mSO(W,S), m\alpha O(W,S), m\beta O(W,S)$, $mBO(W,S), mS_pO(W,S)$ and $mS_\beta O(W,S)$). Additionally, the set comprising all M pre-closed (resp. M semi-closed, Mα-closed, Mβ-closed, M$\dot{b}$-closed, MS_p closed and MS_β - closed) sets denoted by $mPC(W,S)$ (resp. $mSC(W,S)mPC(W,S), m\alpha C(W,S), m\beta C(W,S), mBC(W,S), mS_pC(W,S)$ and $mS_\beta C(W,S)$).

Theorem 2.7. [3] Any union of M pre-open sets remains M pre-open set.

Theorem 2.8. Let $(W, \mathcal{T}_{\mathcal{R}}(S), \mu_{\mathcal{R}}(S))$ be M.T.S.. Then,

1. Every M-open (resp. Msemi-open, Mpre-open, M α-open and M $\dot{b}$-open) set constitutes Mβ-open set [5].
2. Every MS_β-open set constitutes Mβ-open (resp. Msemi-open and M $\dot{b}$-open) set [8].
3. Every MS_P-open set constitutes MS_β-open set [8].

Remark 2.9. None of the converse statements associated with the items in Theorem 2.8 are necessarily true.

Theorem 2.10. [6] Each M pre-open set constitutes M $\dot{b}$-open set.

Remark 2.11. According to [5], the implication relations among these sets are shown in the following diagram:

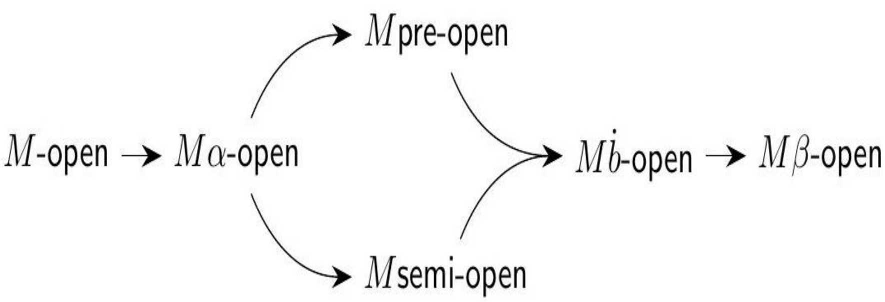

Figure -1 Relationship between the kinds of micro open sets.

3. Micro P_β- open sets

Definition 3.1. An M pre-open set G of an M.T.S. $(W, \mathcal{T}_{\mathcal{R}}(S), \mu_{\mathcal{R}}(S))$ be designated as a Micro P_β-open set, denoted by MP_β-open, provided that for every $x \in G$, we can find a Mβ – closed subset F satisfying $x \in F \subseteq G$. The set containing all MP_β-open sets is denoted by $mP_\beta O(W, S)$.

Remark 3.2. A set is defined as Micro P_β-closed (denoted by MP_β - closed) if it is the complement of an MP_β-open set. This family comprising every such MP_β - closed sets is denoted by $mP_\beta C(W, S)$.

Example 3.3. Consider $W = \{1,2,3\}$ with $W/\mathcal{R} = \{\{1\},\{2,3\}\}$ and $S = \{1,2\} \subseteq W$, then $\mathcal{T}_{\mathcal{R}}(S) = \{W, \emptyset, \{1\}, \{2,3\}\}$ if $\mu = \{2\}$, then $\mu_{\mathcal{R}}(S) = \{W, \emptyset, \{1\}, \{2\}, \{1,2\}, \{2,3\}\} = mPO(W,S) = m\beta O(W,S)$ and $m\beta C(W,S) = \{W, \emptyset, \{1\}, \{3\}, \{1,3\}, \{2,3\}\}$, then $mP_\beta O(W,S) = \{W, \emptyset, \{1\}, \{2,3\}\}$.

Example 3.4. Consider $W = \{a, b, c, d\}$ with $W/\mathcal{R} = \{\{a\},\{b\},\{c,d\}\}$ and $S = \{a, d\} \subseteq W$, then $\mathcal{T}_{\mathcal{R}}(S) = \{W, \emptyset, \{a\}, \{a,c,d\}, \{c,d\}\}$ if $\mu = \{c\}$, then $\mu_{\mathcal{R}}(S) = \{W, \emptyset, \{a\}, \{c\}, \{a,c\}, \{c,d\}, \{a,c,d\}\}$, $mPO(W,S) = \{W, \emptyset, \{a\}, \{c\}, \{a,c\}, \{c,d\}, \{a,b,c\}\}, \{a,c,d\}\}$,

$m\beta O(W,S) = \{W, \emptyset, \{a\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{c,d\}, \{a,b,c\}, \{a,c,d\}, \{b,c,d\}\}$ and $m\beta C(W,S) = \{W, \emptyset, \{a\}, \{b\}, \{d\}, \{a,b\}, \{a,d\}, \{b,d\}, \{c,d\}, \{a,b,d\}, \{b,c,d\}\}$, then $mP_\beta O(W,S) = \{W, \emptyset, \{a\}, \{c,d\}, \{a,c,d\}\}$.

Proposition 3.5. For any given M.T.S, we have the following results:

1. Each MP_β-open set is necessarily Mpre-open set.
2. Each MP_β-open set is necessarily Mβ-open set.
3. Each MP_β-open set is necessarily M$\dot{b}$-open set.

Proof.

1. Results directly from Definition 3.1..
2. Assume G is any MP_β-open set. By Proposition 3.5 (1), it follows that G is M pre-open set. According to Theorem 2.8 (1), which guarantees that every M pre-open set is always an Mβ-open set, we directly conclude that G is an Mβ-open set.
3. Let G be an MP_β-open set. By the result of Proposition 3.5 (1), we obtain G is M pre-open set. By Theorem 2.10, which establishes that the class of M pre-open sets is contained within the class of M$\dot{b}$-open sets, it follows that G is an M$\dot{b}$-open set. □

Remark 3.6.1.

1. Proposition 3.5 (1) is not reversible. Example 3.3 illustrates this fact by demonstrating that , given $mPO(W,S) = \{W, \emptyset, \{1\}, \{2\}, \{1,2\}, \{2,3\}\}$, the set {2} satisfies the Mpre-open condition but is not MP_β-open.
2. Reversing Proposition 3.5 (2) does not always hold valid. As demonstrated by Example 3.4, $m\beta O(W,S) =$

$\{W, \emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}\}$, this set $\{c\}$ satisfies the Mβ-open condition but is not MP_β-open.

3. Reversing Proposition 3.5 (3) does not always hold valid. As evidenced by Example 3.4, $mḃO(W, S) = \{W, \emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}\}$, the set $\{c\}$ satisfies the Mḃ-open condition but is not MP_β-open.
4. The concept of MP_β-open sets and M-open (resp. Msemi-open and M α-open) sets is independent in general. From next Example 3.7 $\{p\}$ represents MP_β-open yet is not M-open (resp. Msemi-open and M α-open) and from Example 3.3, $\mu_{\mathcal{R}}(S) = \{W, \emptyset, \{1\}, \{2\}, \{1,2\}, \{2,3\}\} = mSO(W, S) = m\alpha O(W, S)$, we can easily observe that the set $\{2\}$ is M-open (resp. M semi-open and Mα-open) set , yet it fails to be MP_β-open set.
5. The concept of MP_β-open sets and MS_β-open (resp. MS_P-open) sets is independent in general. From next Example 3.7, $\{p\}$ is MP_β-open but not MS_β-open (resp. MS_P-open) and from Example 3.4, $mS_\beta O(W, S) = \{W, \emptyset, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$ and $mS_P O(W, S) = \{W, \emptyset, \{a, b\}, \{b, c, d\}\}$, it is clear that the set $\{a, b\}$ represents MS_β-open (resp. MS_P-open) set, but not MP_β-open .

Example 3.7. Consider $W = \{p, q, r\}$ with $W/\mathcal{R} = \{\{p, q\}, \{r\},\}$ and $S = \{p, q\} \subseteq W$, then $\mathcal{T}_{\mathcal{R}}(S) = \{W, \emptyset, \{p, q\}\}$ if $\mu = \{r\}$, then $\mu_{\mathcal{R}}(S) = \{W, \emptyset, \{r\}, \{p, q\}\} =, mSO(W, S) = m\alpha O(W, S) = mS_p O(W, S) = mS_\beta O(W, S)$, $mPO(W, S) = m\beta O(W, S) = m\beta C(W, S) = \{W, \emptyset, \{p\}, \{q\}, \{r\}, \{p, q\}, \{p, r\}, \{q, r\}\}$, then $mP_\beta O(W, S) = \{W, \emptyset, \{p\}, \{q\}, \{r\}, \{p, q\}, \{p, r\}, \{q, r\},\}$.

Remark 3.8. Based on the findings in Remark 2.11, Theorem 2.8, and Proposition 3.5, we can construct the subsequent implication diagram:

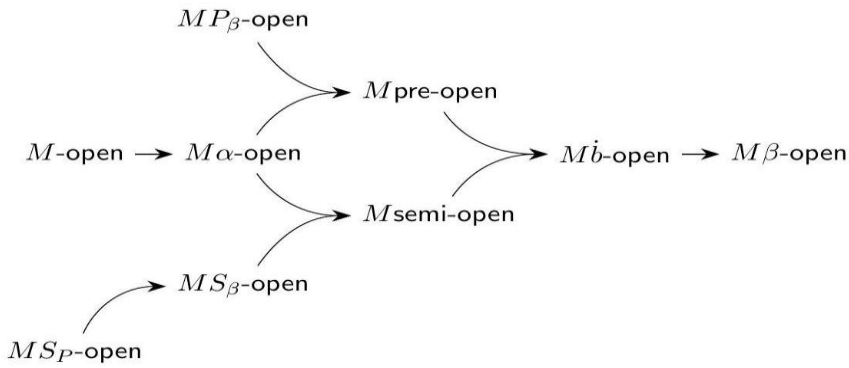

Figure -2 Relationship between the different kinds of micro open sets and MP_β-open sets.

Proposition 3.9. If G is contained in W, while H is MP_β-open subset of W fulfilling $H \subseteq G \subseteq Mcl(Mint(H))$. Then it follows that G represents Mβ-open in W.

Proof. Let H be an MP_β-open set in $(W, \mathcal{T}_{\mathcal{R}}(S), \mu_{\mathcal{R}}(S))$, By theorem 3.5 (1), H is a Mpre-open, then $H \subseteq Mint(Mcl(H))$. Now $G \subseteq Mcl(Mint(H)) \subseteq Mcl(Mint(Mint(Mcl(H)))) = Mcl(Mint(Mcl(H))) \subseteq Mcl(Mint(Mcl(G)))$. Hence $G \subseteq Mcl(Mint(Mcl(G)))$. Therefore G is $M\beta$-open in W. □

Theorem 3.10. The union of any collection of MP_β-open sets remains a MP_β-open set.

Proof. Let $\{G_i; i \in \Lambda\}$ stand for a family comprising MP_β-open sets, by Proposition 3.5 (1), the sets $\{G_i; i \in \Lambda\}$ are M pre-open set, and according to Theorem 2.7, for each $x \in \cup G_i \subseteq mPO(W, S)$, one can find a β - closed set F satisfying $x \in F \subseteq G_i$. Consequently $x \in F \subseteq \cup G_i$. Therefore, $\cup \{G_i; i \in \Lambda\}$ is also MP_β-open set. □

Theorem 3.11. The intersection of any arbitrary collection of $MP_\beta -$ closed sets results in an $MP_\beta -$ closed set.

Proof. Let $\{F_i; i \in \Lambda\}$ be a collection of MP_β-closed sets. Let $G_i = (F_i)^C$, hence $\{G_i; i \in \Lambda\}$ be a family of MP_β-open sets. According to Theorem 3.10, $\cup G_i = \cup (F_i)^C$ becomes a MP_β-open set. Consequently $(\cap F_i)^C$ is MP_β-open set. This leads to $(\cap F_i)$ being an MP_β-closed set. □

Remark 3.12. The property of being MP_β-open is not necessarily preserved under finite intersections. This is illustrated in Example 3.13, where $\{i, j, k\}, \{i, j, l\} \in mP_\beta O(W, S)$ but $\{i, j, k\} \cap \{i, j, l\} = \{i, j\} \notin mP_\beta O(W, S)$.

Example 3.13. Consider $W = \{i, j, k, l\}$ with $W/\mathcal{R} = \{\{i, j\}, \{k\}, \{l\}\}$ and $S = \{k, l\} \subseteq W$, then $\mathcal{T}_\mathcal{R}(S) = \{W, \emptyset, \{k, l\}\}$ if $\mu = \{i\}$, then $\mu_\mathcal{R}(S) = \{W, \emptyset, \{i\}, \{k, l\}, \{i, k, l\}\}$, $mPO(W, S) = \{W, \emptyset, \{i\}, \{k\}, \{l\}, \{i, k\}, \{i, l\}, \{k, l\}, \{i, j, k\}, \{i, j, l\}, \{i, k, l\}\}$, $m\beta O(W, S) = \{W, \emptyset, \{i\}, \{k\}, \{l\}, \{i, j\}, \{i, k\}, \{i, l\}, \{j, k\}, \{j, l\}, \{k, l\}, \{i, j, k\}, \{i, j, l\}, \{i, k, l\}, \{j, k, l\}\}$ and $m\beta C(W, S) = \{W, \emptyset, \{i\}, \{j\}, \{k\}, \{l\}, \{i, j\}, \{i, k\}, \{i, l\}, \{j, k\}, \{j, l\}, \{k, l\}, \{i, j, k\}, \{i, j, l\}, \{j, k, l\}\}$, then $mP_\beta O(W, S) = \{W, \emptyset, \{i\}, \{k\}, \{l\}, \{i, k\}, \{i, l\}, \{k, l\}, \{i, j, k\}, \{i, j, l\}, \{i, k, l\}\}$.

Definition 3.14. In an M.T.S. ($W, \mathcal{T}_\mathcal{R}(S), \mu_\mathcal{R}(S)$), an element $x \in A$ is a Micro P_β - interior point of A provided that $x \in G \subset A$ for some MP_β-open set G, and the collection of all such Micro P_β - interior points constitutes the Micro P_β - interior of A, denoted as $MP_\beta\text{int}(A)$.

Theorem 3.15. The properties listed below are valid for any arbitrary subset A of U:

1. $MP_\beta\text{int}(\emptyset) = \emptyset$.
2. $MP_\beta\text{int}(W) = W$.
3. $MP_\beta\text{int}(A) \subseteq A$.
4. If $A \subseteq B$, then $MP_\beta\text{int}(A) \subseteq MP_\beta\text{int}(B)$.
5. $MP_\beta\text{int}(A)$ is equal to the union of all MP_β-open set contained in A.
6. If A is a MP_β-open set then $A = MP_\beta\text{int}(A)$.
7. $MP_\beta\text{int}(A \cup B) \supseteq MP_\beta\text{int}(A) \cup MP_\beta\text{int}(B)$.
8. $MP_\beta\text{int}(A \cap B) \subseteq MP_\beta\text{int}(A) \cap MP_\beta\text{int}(B)$.
9. $MP_\beta\text{int}[MP_\beta\text{int}(A)] = MP_\beta\text{int}(A)$.

Proof. The proof of (1), (2) and (3) follow from Definition 3.14

4. Suppose $A \subseteq B$. We need to prove that $MP_\beta\text{int}(A) \subseteq MP_\beta\text{int}(B)$.Let $x \in MP_\beta\text{int}(A)$. From Definition 3.14, one can find a MP_β-open set G satisfying $x \in G \subseteq A$. Since $A \subseteq$

B, it follows that MP_β-open set G such that $x \in G \subseteq B$. from Definition 3.14, $x \in MP_\beta int(B)$. That is $MP_\beta int(A) \subseteq MP_\beta int(B)$.

5. To verify the equality $MP_\beta int(A) = \cup \{G: G \subset A, G \text{ is } MP_\beta\text{-open set}\}$, that an element x belongs to $MP_\beta int(A)$. Consequently, we can find a MP_β-open set G satisfying $x \in G \subset A$. Hence $x \in \cup \{G: G \subset A, G \text{ is } MP_\beta\text{-open set}\}$. For the other direction, assume $x \in \cup \{G: G \subset A, G \text{ is } MP_\beta\text{-open set}\}$, it follows that some set $G_o \subset A$ exists providing $x \in G_o$, with G_o being MP_β-open set, i.e., $x \in MP_\beta int(A)$. Hence $\cup \{G: G \subset A, G \text{ is } MP_\beta\text{-open set}\} \subset MP_\beta int(A)$. Therefore, $MP_\beta int(A) = \cup \{G: G \subset A, G \text{ is } MP_\beta\text{-open set}\}$.
6. Let A be an MP_β-open set. Our aim will be to show that $MP_\beta int(A) = A$. Let $x \in A$ that is clear $x \in A \subseteq A$. Since A is a MP_β-open set and from Definition 3.14, $x \in MP_\beta int(A)$, we get $A \subseteq MP_\beta int(A)$ and from part (3) we get $MP_\beta int(A) = A$.
7. It is clear $A \subseteq (A \cup B)$ and $B \subseteq (A \cup B)$. By part (4), $MP_\beta int(A) \subseteq MP_\beta int(A \cup B)$ and $MP_\beta int(B) \subseteq MP_\beta int(A \cup B)$ implies $MP_\beta int(A) \cup MP_\beta int(B) \subseteq MP_\beta int(A \cup B)$.
8. Let $x \in MP_\beta int(A \cap B)$. This implies the existence of a MP_β-open set G, satisfying $x \in G \subseteq A \cap B)$. That is we can find a MP_β-open set, satisfying $x \in G \subseteq A$ and $x \in G \subseteq B$. Hence $x \in MP_\beta int(A)$ and $x \in MP_\beta int(B)$. That is $x \in MP_\beta int(A) \cap MP_\beta int(B)$. Thus $MP_\beta int(A \cap B) \subseteq MP_\beta int(A) \cap MP_\beta int(B)$.
9. According to part (5), $MP_\beta int(A)$ can be defined as the union of every $MP_\beta int(A)$ set lying within A. This means that $MP_\beta int(A)$ is an MP_β-open set. Hence by part (6) $MP_\beta int[MP_\beta int(A)] = MP_\beta int(A)$.

Remark 3.16. Regarding parts (4), (7) and (8) of Theorem 3.15, their converses are not necessarily valid, which is demonstrated in the following instance.

Example 3.17. Based on Example 3.13, $mP_\beta O(W,S) = \{W, \emptyset, \{i\}, \{k\}, \{l\}, \{i,k\}, \{i,l\}, \{k,l\}, \{i,j,k\}, \{i,j,l\}, \{i,k,l\}\}$. Let $A = \{j\}$, $B = \{i,k,l\}$, $C = \{i,j,k\}$ and $D = \{i,j,l\}$, then

4. Since $MP_\beta int(A) = \emptyset$ and $MP_\beta int(B) = \{i,k,l\} = B$, it is clear that $MP_\beta int(A) \subseteq MP_\beta int(B)$, but $A \nsubseteq B$.
5. Since $A \cup B = W$, we have $MP_\beta int(A \cup B) = W$. On the other hand, $MP_\beta int(A) \cup MP_\beta int(B) = \{i,k,l\}$, hence $MP_\beta int(A \cup B) \nsubseteq MP_\beta int(A) \cup MP_\beta int(B)$.
6. Since $MP_\beta int(C) = \{i,j,k\} = C$ and $MP_\beta int(D) = \{i,j,l\} = D$, and noting that $C \cap D = \{i,j\}$, we have $MP_\beta int(C \cap D) = \{i\}$. On the other hand, $MP_\beta int(C) \cap MP_\beta int(D) = \{i,j\}$. Hence $MP_\beta int(C) \cap MP_\beta int(D) \nsubseteq MP_\beta int(C \cap D)$.

Definition 3.18. ($W, \mathcal{T}_\mathcal{R}(S), \mu_\mathcal{R}(S)$) be a M.T.S.. and $A \subset W$. Micro P_β - closure of A is represented by the intersection of every MP_β - closed supersets of A. This operator is denoted by $MP_\beta cl(A)$, where:

$MP_\beta cl(A) = \cap \{F: F \supset A, F \text{ is } MP_\beta - closed \text{ set}\}$.

Theorem 3.19. Let A and B be subsets of ($W, \mathcal{T}_{\mathcal{R}}(S), \mu_{\mathcal{R}}(S)$). Then

1. $\mathrm{MP_\beta cl}(\emptyset) = \emptyset$.
2. $\mathrm{MP_\beta cl}(W) = W$.
3. $A \subseteq \mathrm{MP_\beta cl}(A)$.
4. $\mathrm{MP_\beta cl}(A)$ is $\mathrm{MP_\beta} -$ closed set.
5. If $A \subseteq B$, then $\mathrm{MP_\beta cl}(A) \subseteq \mathrm{MP_\beta cl}(B)$.
6. $\mathrm{MP_\beta cl}(A) \cup \mathrm{MP_\beta cl}(B) \subseteq \mathrm{MP_\beta cl}(A \cup B)$.
7. $\mathrm{MP_\beta cl}(A \cap B) \subseteq \mathrm{MP_\beta cl}(A) \cap \mathrm{MP_\beta cl}(B)$.
8. The equality $\mathrm{MP_\beta cl}(A) = A$ holds true if, and only if, the set A is $\mathrm{MP_\beta}$-closed.
9. $\mathrm{MP_\beta cl}[\mathrm{MP_\beta cl}(A)] = \mathrm{MP_\beta cl}(A)$.

Proof. The proof of (1), (2), (3), (4) and (5) follows from Definition 3.18

6. we need prove, $\mathrm{MP_\beta cl}(A) \cup \mathrm{MP_\beta cl}(B) \subseteq \mathrm{MP_\beta cl}(A \cup B)$. It is clear that $A \subseteq A \cup B$ and $B \subseteq A \cup B$. By part (5), $\mathrm{MP_\beta cl}(A) \subseteq \mathrm{MP_\beta cl}(A \cup B)$ and $\mathrm{MP_\beta cl}(B) \subseteq \mathrm{MP_\beta cl}(A \cup B)$, then
$\mathrm{MP_\beta cl}(A) \cup \mathrm{MP_\beta cl}(B) \subseteq \mathrm{MP_\beta cl}(A \cup B)$.
7. To prove, $\mathrm{MP_\beta cl}(A \cap B) \subseteq \mathrm{MP_\beta cl}(A) \cap \mathrm{MP_\beta cl}(B)$. We know that $A \cap B \subseteq A$ and $A \cap B \subseteq B$. By part (5), $\mathrm{MP_\beta cl}(A \cap B) \subseteq \mathrm{MP_\beta cl}(A)$ and $\mathrm{MP_\beta cl}(A \cap B) \subseteq \mathrm{MP_\beta cl}(B)$, then $\mathrm{MP_\beta cl}(A \cap B) \subseteq \mathrm{MP_\beta cl}(A) \cap \mathrm{MP_\beta cl}(B)$.
8. From Definition 3.18, $\mathrm{MP_\beta cl}(A) = \cap \{F : F \supset A, F \text{ is } \mathrm{MP_\beta} - \text{closed set}\}$.If A is $\mathrm{MP_\beta}$ -closed set then A is a member of the above collection and each member contains A. Hence their intersection is A and $\mathrm{MP_\beta cl}(A) = A$. Conversely, if $\mathrm{MP_\beta cl}(A) = A$. By part (4) $\mathrm{MP_\beta cl}(A)$ is $\mathrm{MP_\beta}$-closed set, then A is $\mathrm{MP_\beta}$ - closed set.
9. We need to prove $\mathrm{MP_\beta cl}[\mathrm{MP_\beta cl}(A)] = \mathrm{MP_\beta cl}(A)$.By part (4) $\mathrm{MP_\beta cl}(A)$ forms a $\mathrm{MP_\beta}$-closed set. By part (8), we have $\mathrm{MP_\beta cl}(A)$ is $\mathrm{MP_\beta}$-closed if and only if $\mathrm{MP_\beta cl}[\mathrm{MP_\beta cl}(A)] = \mathrm{MP_\beta cl}(A)$.

Remark 3.20. The converse of part (5), (6) and (7) in Theorem 3.19 fails to be always valid, as illustrated by the subsequent instance.

Example 3.21. From Example 3.13, $mP_\beta O(W,S) = \{W, \emptyset, \{i\}, \{k\}, \{l\}, \{i,k\}, \{i,l\}, \{k,l\}, \{i,j,k\}, \{i,j,l\}, \{i,k,l\}\}$. and we obtain $mP_\beta C(W,S) = \{W, \emptyset, \{j\}, \{k\}, \{l\}, \{i,j\}, \{j,k\}, \{j,l\}, \{i,j,k\}, \{i,j,l\}, \{j,k,l\}\}$. Let $A = \{j\}$, $B = \{i\}$, $C = \{l\}$, and $D = \{k\}$.

5. Since $\mathrm{MP_\beta cl}(A) = \{j\} = A$ and $\mathrm{MP_\beta cl}(B) = \{i,j\}$, it is clear that $\mathrm{MP_\beta cl}(A) \subseteq \mathrm{MP_\beta cl}(B)$, but $A \nsubseteq B$.

6. Noting that $MP_{\beta}cl(C) = \{l\} = C$ and $MP_{\beta}cl(D) = \{k\}$. Noting that $C \cup D = \{k, l\}$, we get $MP_{\beta}cl(C \cup D) = \{j, k, l\}$. On the other hand, $MP_{\beta}cl(C) \cup MP_{\beta}cl(D) = \{k, l\}$. Hence $MP_{\beta}cl(C \cup D) \nsubseteq MP_{\beta}cl(C) \cup MP_{\beta}cl(D)$.

7. Since $A \cap B = \emptyset$, we have $MP_{\beta}cl(A \cap B) = \emptyset$. On the other hand, $MP_{\beta}cl(A) \cap MP_{\beta}cl(B) = \{j\}$. Hence $MP_{\beta}cl(A) \cap MP_{\beta}cl(B) \nsubseteq MP_{\beta}cl(A \cap B)$.

4. Conclusion

In conclusion, this study successfully introduces and establishes the topological behavior of Micro P_{β} - open sets. Based on the proven theorems, we conclude that this new class is closed under arbitrary unions, yet it diverges from standard topologies as it is not necessarily preserved under finite intersections. Furthermore, the constructed counterexamples explicitly clarify the topological hierarchy; they confirm that Micro P_{β} - open sets form a strictly stronger condition than Micro pre-open, Micro $\dot{b}$-open, and Micro β -open sets, while remaining theoretically independent of classes like Micro semi-open and Micro α-open sets. Consequently, the deduced properties of the newly defined Micro P_{β}-interior and Micro P_{β}-closure operators provide a verified mathematical framework that directly supports our future investigations into micro continuity.

References

[1] M Lellis Thivagar and Carmel Richard. On nano forms of weakly open sets. International journal of mathematics and statistics invention, 1(1):31-37, 2013.

[2] Zdzisław Pawlak. Rough sets. International journal of computer & information sciences, 11(5):341-356, 1982.

[3] Sakkraiveeranan Chandrasekar. On micro topological spaces. Journal of New Theory, (26):23-31, 2019.

[4] S Chandrasekar and G Swathi. Micro- α-open sets in micro topological spaces. International Journal of Research in Advent Technology, 6(10):2633-2637, 2018.

[5] Hariwan Z Ibrahim. Micro β-open sets in micro topology. General letters in Mathematics, 8(1):8-15, 2020.

[6] Taha H. Jasim and Marwah D. Sultan. On micro - b - open sets and micro - b -continuity. Tikrit Journal of Pure Science, 25:124-129, 12 2020.

[7] M Maheswari, S Dhanalakshmi, and N Durgadevi. Micro sp-open sets in micro topological spaces. Ratio Mathematica, 45, 2023.

[8] Mridul Kumar Gupta and Shivangi Tyagi. Generalization of micro-open set and an application in healthcare. Journal of Mountain Research, 19, 2024

[9] Esraa A. Dawood and Y. Y. Yousif. Exploring generalizations of fibrewise micro-topological spaces and their applications in mathematics. AIP Conference Proceedings, 3169:070053, 2025.

[10] Esraa A. Dawood and Y. Y. Yousif. Fibrewise micro-topological spaces. *AIP Conference Proceedings*, 3169:070023, 2025.

[11] E. A. Dawood and Y. Y. Yousif. Fibrewise micro-connected topological spaces. *Journal of Interdisciplinary Mathematics*, 29(1):129-135, 2026.