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From Polynomial Ring to α-ideal Topological Space

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Abstract

The reduction of a polynomial (poly) over Galois Fields (GFs) reveals whether the coefficients of that poly truly belong to GFs, based on their irreducibility into non-trivial polys. $GF(2^n)$ is constructed from the primitive poly $u^m + u + 1$ through the ring$Z_2[u]/\langle u^m + u + 1\rangle$. Furthermore, this study is related to the framework of ideal topological spaces, where a binary relation is derived through the algebraic structure represented by the u-adic valuation function $v_u(p)$. Starting from the equivalence classes derived from the valuation based relation, a partition topology is generated on the finite field. Combining the ideal I with the topological structure enables us to define both the closure operator cl_α^* and the α-local function. The significance of these two operators becomes evident through their ability to classify field elements with high precision, there by revealing the deep structure of finite fields. Furthermore, the α-ideal topology provides a more refined partition of the field elements, the filtration of negligible elements through the ideal, and a systematic study of the structural properties of polynomials.

Keywords: Galois Field $GF(2^n)$, Irreducible Polynomials, u-adic Valuation, Equivalence Relation, Ideal Topological Spaces, α-open Sets, α-Local function, Closure Operator ($\boldsymbol{cl_\alpha^*}$).

من حلقات كثيرة الحدود الى الفضاء التبولوجي المثالي لـ $\boldsymbol{\alpha}$

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الخلاصة

يكشف اختزال كثيرة الحدود (poly) فوق حقول غالوا (GFs) ما إذا كانت معاملات تلك الكثيرة تنتمي فعلاً إلى GFs، وذلك استناداً إلى قابليتها للتحليل إلى كثيرات حدود غير تافهة. يُبنى $GF(2^{\mathrm{n}})$من كثيرة الحدود الأولية $u^{\mathrm{m}} + u + 1$ عبر الحلقة $Z_2[u]/(u^{\mathrm{m}} + u + 1)$. علاوةً على ذلك، يرتبط هذا البحث بإطار الفضاءات التبولوجية المثالية، حيث تُستخرج علاقة ثنائية من خلال البنية الجبرية التي تمثلها دالة التقييم u-ادية $v_{\mathrm{u}}(p)$. انطلاقاً من فئات التكافؤ المشتقة من علاقة التقييم، تُولِّد تبولوجيا تجزئة على الحقل المنتهي. إن الجمع بين المثالي I والبنية التبولوجية يُمكّننا من تعريف كلٍّ من مؤثر الإغلاق cl_α^* والدالة α-الموضعية. تتجلى أهمية هذين المؤثرين من خلال قدرتهما على تصنيف عناصر الحقل المنتهي بدقة عالية، مما يكشف البنية العميقة للحقول المنتهية. فضلاً عن ذلك، توفر تبولوجيا α-المثالية تجزئةً أكثر دقةً لعناصر الحقل، وتصفيةً للعناصر المهملة عبر المثالي، ودراسةً منهجيةً للخصائص البنيوية لكثيرات الحدود.

1. Introduction

The study of algebraic structures over finite fields, particularly Galois fields $GF(2^n)$, has long been a cornerstone of modern mathematics due to its extensive applications in coding theory and cryptography [1]. A fundamental approach to understanding these fields is through the analysis of polynomial rings $Z_2[u]/\langle u^m + u + 1\rangle$. The structural properties of polynomials within these rings can be effectively characterized using the u-adic valuation function $v_u(p)$, which serves as a pivotal tool for partitioning algebraic elements into distinct equivalence classes.

In 1930, Kuratowski laid the foundation for the concept of ideal topological spaces, which has gained widespread interest in general topology. Vaidyanathaswamy subsequently deepened and developed the theoretical foundations of this concept [12]. To understand how to manage negligible sets and classify the elements of essential sets, we employ the concept of the ideal I on (X, τ) as a comprehensive methodological framework, as well as providing a framework for handling negligible sets, leading to a more precise and comprehensive analysis. This field has witnessed significant knowledge accumulation, beginning with Abd-El-Monsef [13] and his team, who opened a new avenue through the s-local function, followed by the contributions of Khan and Noiri [14], and culminating in the local function A^*, shedding light on its role in generating finer and more refined topological spaces. Recent studies have contributed to enriching this field of knowledge, Yalaz and Kaymakcı [19] introduced the concept of a new local function along with innovative compatibility types within ideal topological spaces, while Al-Omeri and Noiri [21] pursued this trajectory by investigating novel classes of open sets within the same framework. Building upon these developments, this research aims to establish a clear and solid connection between two worlds: the algebraic world, represented by polynomial rings, and the topological world, represented by ideal spaces. This is achieved by adopting the equivalence classes derived from the valuation function $v_u(p)$ to construct a partition topology on $GF(2n)$. Furthermore, this research expands to encompass the utilization of α-open sets as a pivotal tool for defining the α-local function, as well as defining the closure operator.

The integration of the topological ideal I with the α-ideal structure forms the fundamental concept for constructing the α-ideal topology. What distinguishes this approach is its ability to classify field elements, isolate negligible algebraic structures, and filter them with a high degree of precision. The α-ideal topology demonstrates its efficiency in differentiating between essential and secondary elements within the finite field $GF(2^n)$, thereby opening new pathways toward a deeper understanding of the internal structure of these fields.

Despite the significance of these contributions, a careful review of the literature reveals a fundamental research gap: the theoretical literature lacks a study that directly links the algebraic structure of finite fields $GF(2^n)$ to the construction of topological spaces of the α-ideal type through the employment of the u-adic valuation function. Previous efforts have focused either on treating the algebraic properties of polynomial rings independently, or on formulating ideal topological spaces that lack a solid foundation connecting them to the structure of a concrete finite field. In contrast, this work proposes a solution to this gap by formulating a partition topology on $GF(2^n)$ derived from the equivalence classes of the valuation function, thereby paving the way for the construction of a topological space of the α-ideal type that precisely reveals the deep structural components of finite fields.

2. Preliminaries and Mathematical Background:

This section addresses the fundamental concepts of α-ideal topological spaces and Galois fields, where the α-ideal topological space is defined by the symbol (X, τ_{α}^{*}). The set X is composed of elements derived from the field $GF(2^n)$ via an irreducible polynomial, while the construction of the topology is based on the collection of α-open sets coupled with the ideal I.

Definition 2.1 [2]. A field is defined as a commutative ring F where $1 \neq 0$, and for every element $a \neq 0$ in F there exists a multiplicative inverse a^{-1}, satisfying the condition $a \cdot a^{-1} = a^{-1} \cdot a = 1$.

Definition 2.2 [3]. The polynomial ring $R[u]$ over R consists of all polynomials in u with coefficients in R, and is given in the following form:

$$f(u) = \sum_{i=0}^{n} a_i u^i = a_0 + a_1 u + a_2 u^2 + \ldots\ldots + a_n u^n \qquad (1)$$

It is worth noting here that ai belongs to R for every i in this context, with the condition that the number of non-zero coefficients remains finite.

Example 2.1. For $R = Z_2$, the polynomial ring is $Z_2[u] = \{a_0 + a_1 u + a_2 u^2 + \cdots + a_n u^n \mid a_i \in \{0,1\}$.

Definition 2.3 [2]. A single-variable poly $p(u) \in F[u]$ is called irreducible over F if deg $p(u) \geq 1$ and it cannot be factored in the form of two polys $g(u), h(u) \in F[u]$, each having degree smaller than deg $(p(u))$.

Example 2.2. For illustration purposes, consider the polynomial $f(u) = u^4 + u^3 + u^2 + u + 1 \in Z_2[u]$.

where it can be verified that it is irreducible over Z_2 and cannot be factored into a product of lower degree factors.

Definition 2.4 [4]. The field F is termed a finite field if it contains a finite number of elements.

Definition 2.5 [7]. The mathematical structure $GF(p^n)$ is based on the definition of Galois fields with cardinality p^n. In this context, p represents the characteristic prime of the field, while n represents the value of positive integers. In the special case $p = 2$, the field is written as $GF\ (2^n)$. Moreover, a poly is called monic when its leading coefficient is equal to 1, and the term containing the highest power of u, namely u^n, is referred to as the leading term.

Remark 2.1. A polynomial is monic if its leading coefficient is 1.

Theorem 2.1 [5]. Consider a prime p and a monic, irreducible poly ring $p(u)$ of degree n. As a result, from the field of order p^n the quotient ring is formed.

The reader may refer to [5] for further details regarding the proof.

Theorem 2.2 [6]. An equivalence relation produces a unique partition of the elements for that set.

Definition 2.6 [6]. The ordered pair (X, τ) constitutes a topological space, where τ comprises a family of subsets of X. The formation of a valid topology requires three conditions to be satisfied: that both $\emptyset$ and X belong to τ, that τ contains any arbitrary union of its elements, and that every finite intersection of its sets remains within τ.

Definition 2.7[8] Let (X, τ) be a topological space and let $A \subseteq X$. We define certain types of sets as follows:

- A subset $A \subseteq X$ is called preopen if it satisfies the condition: A⊆ int(cl(A)).
- A subset $A \subseteq X$ is considered an α-open set if it satisfies the condition: A⊆ int(cl(int(A))).

Remark 2.2 [14] In a topological space(X,τ), a set A is said to be α-closed if and only if its complement A^c is an α-open set.

Definition 2.8 [9] A non-empty collection $I \subseteq P(X)$ is considered an ideal on (X,τ) if the following conditions are satisfied:

1. $\emptyset \in I$.
2. If $B_1 \in I$ and $B_2 \subseteq A$, then $B_1 \in I$.
3. If $B_1, B_2 \in I$, then $B_1 \cup B_2 \in I$.

It is worth noting here that (X,τ,I) represents an ideal topological space.

Definition 2.9 [9][16] The local function of a subset $A \subseteq X$ with respect to the ideal I and the topology τ is defined as follows:

$$A^*(I) = \{u \in X \mid U \cap A \notin I, \forall U \in \tau(u)\} \qquad (2)$$

where $\tau(u)$ comprises all open sets containing u. For simplicity, the notation $A^*(I)$ is frequently abbreviated as A^*.

Definition 2.10 [10] In the ideal topological space (X,τ,I), the Kuratowski closure operator $cl^*(A)$ for any subset $A \subseteq X$ is defined as the union of the set A with its local function, expressed as $cl^*(A) = A \cup A^*$. This operator generates a topology given by the formula:

$$\tau*(I,\tau) = \{A \subseteq X : cl*(Ac) = Ac\} \qquad (3)$$

which is finer than the original topology τ in the general case. Furthermore, a subset $A \subseteq X$ is considered a τ^*-closed set if and only if the condition $A^* \subseteq A$ holds.

Definition 2.11 [17][18] The α-open sets form a topological structure on X in the topological space (X,τ), known as τ_α, and is defined as follows:

$$\tau\alpha(u) = \{U \in \tau\alpha : u \in U\} \qquad (4)$$

Definition 2.12 [11][15] In the ideal topological space (X,τ,I), the α-local function of any subset $A \subseteq X$ with respect to the ideal I and τ_α is defined as follows:

$$A^*_\alpha(I,\tau) = \{x \in X : A \cap U \notin I, U \in \tau_\alpha(x)\} \qquad (5)$$

$\tau_\alpha(u)$ represents the collection of α-open sets containing u. To simplify the notation, is frequently used as a shorthand for $A^*_\alpha(I,\tau)$.

3. α-Ideal Topological Space Structures Derived by Valuation Relation on $GF(2^n)$

Based on the fundamental concepts of polys and Galois fields, we begin this study through the quotient ring $Z_2[u]/\langle u^m + u + 1\rangle \cong GF(2^n)$, where m represents the deg of the irreducible poly and n represents the deg of field extension. We transition from the algebraic structure to the α-ideal topological space through a binary relation derived from the poly valuation. For any polynomial p(u) = $a_0 + a_1$u + $a_2u^2 + \cdots + a_{n-1}u^{n-1}$ in the field $GF(2^n)$, where the coefficients $a_i \in \{0,1\}$, we define the valuation function $v_u(p)$ as the index of the lowest non-zero coefficient:

$$v_u(p) = min\{i \mid a_i \neq 0\} \qquad (6)$$

For the additive identity (zero element), we define $v_u(0) = \infty$. The study is based on the following equivalence relation R:

$$(p,k) \in R \Leftrightarrow v_u(p) = v_u(k) \qquad (7)$$

By using the equivalence classes resulting from this valuation relation as a fundamental basis for constructing a topology, we transition from algebraic operations to an integrated topological. The topological space allows the study of proximity and connectivity in the field $GF(2^n)$ based on the structural alignment of the polys coefficients. In this study, Within the framework of our study on the construction of the ideal topological space (X, τ, I), we employ α-open sets to define the α-local function on subsets $A \subseteq X$ in the following manner:

$$A^*_\alpha(I, \tau) = \{u \in X : A \cap U \notin I \ for\ every\ U \in \tau_\alpha(u)\} \quad (8)$$

The α-star closure operator is then induced as, which forms the basis for the refined topology. This transition allows for a robust analysis of the field $GF(2^n)$ by filtering out negligible algebraic structures through the topological ideal.

Theorem 3.1. The u-adic valuation defined on the field $GF(2^n)$ by $v_u(p) = min\{i \mid a_i \neq 0\}$ produce an equivalence relation.

proof. To prove that $v_u(p)$ yield equivalence relation R, the following three conditions must be satisfied:

1.Reflexivity: For any polynomial $p \in GF(2^n)$, $v_u(p) = v_u(p)$. Therefore $(p, p) \in R$ for all $p \in X$.

2.Symmetry: Suppose $(p, k) \in R$. By definition, this implies that $v_u(p) = v_u(k)$. Since equality is symmetric in the set of integers $N \cup \{0, \infty\}$, it follows that $v_u(k) = v_u(p)$, which means $(k, p) \in R$.

3.Transitivity: Suppose $(p, k) \in R$ and $(k, h) \in R$. This implies that $v_u(p) = v_u(k)$ and $v_u(k) = v_u(h)$. By the transitive property of equality, we conclude that $v_u(p) = v_u(h)$, which means $(p, h) \in R$.

By satisfying the required axioms, it becomes evident that the function $v_u(p)$ represents an equivalence relation on the field $GF(2^n)$, which leads to the decomposition of the field into sets known as equivalence classes, denoted by the symbol $[u]_R$.

Theorem 3.2. A finite field $X = GF(2^n)$ is generated by an irreducible polynomial ring, and R is a binary relation on X, defined by the u-adic valuation $v_u(p)$, then R represents a basis for a topological space.

proof. Let X be a Galois field, and the $v_u(p)$ defines an a equivalence relation by Theorem 3.1, and every equivalence relation on X induces a unique partition into disjoint equivalence classes $[u]_R$by Theorem 2.2, Since X is a union of all its equivalence classes, the relation $\cup [u_i]_R = X$, holds, which satisfies the first axiom of the topological basis, requiring that every element of the space belongs to at least one basis element, The intersection of any two classes $[u]_R \cap [u]_R = \emptyset$ (if they are distinct) or equal to the class itself in the case of their identity, which means the second axiom of the topological basis is satisfied (the intersection of any two basis elements represents a union of elements from it). Consequently, the collection $\beta = \{[u_i]_R : u_i \in X\}$ produce a topology on X. Accordingly, β represents a basis for generating a partition topology τ on X, in which the open sets are arbitrary unions of equivalence classes.

Definition 3.1 In the context of the ideal topological space (X, τ, I), and considering any subset $A \subseteq X$, we conclude that:

$$\tau_\alpha^* = \{A \subseteq X : cl_\alpha^*(A^c) = A^c\} \qquad (9)$$

This topology is called α-ideal topological spaces.

Theorem 3.3. In the ideal topological space (X, τ, I) , the topology τ_α^* is finer than τ, as every open set in τ retains its openness within, hence $\tau \subseteq \tau_\alpha^*$.

Proof. Assume $U \in \tau$. To prove that, we consider its complement $F = X - U$, which is closed in τ, implying $cl(F) = F$. Based on the locality property of the α-type function, and given that every accumulation point generated by this function for any closed set F in the topology τ (which naturally satisfies $cl(F) = F$) must lie contained within F, it necessarily follows that: $F_\alpha^* \subseteq cl(F) = F$. Therefore, when computing the α-star closure of F, we obtain, which means that F is α-closed. It follows that every element U taken from the original topology τ is at the same time an element of τ_α^*, which establishes the validity of the inclusion relation $\tau \subseteq \tau_\alpha^*$.

Proposition 3.1 Let X be a Galois Field and P is the partition of X by the relation $v_x(p)$, i.e $P = X/v_x(p)$, which makes a base for some topology τ on X. If I be an ideal includes any one of the sets of P, it is a discrete topology

Proof. To prove that is discrete, we show that for all a ∈ X. This is equivalent to showing that $cl_\alpha^*(\{a\}^c) = \{a\}^c$. Let $A = \{a\}^c = X \setminus \{a\}$. Since, to complete the proof, it suffices to show that: $A_\alpha^* \subseteq A$, Let $x \in A_\alpha^*$. To show that $x \in A$ (i.e., $x \neq a$), assume $x = a$. By the definition of A_α^* , $\forall U \in \tau_\alpha(x)$, $U \cap A \notin I$, Since $P = X/v_x(p)$ is a base for τ, let $U = E_a \in P$ such that a $\in E_a$. Then: $U \cap A = E_a \cap (X \setminus \{a\}) = E_a \setminus \{a\}$. By hypothesis, the ideal I contains at least one entire equivalence class from P. If $E_a \subseteq$ I, then $E_a \setminus \{a\} \in$ I, This implies $U \cap A \in I$, But this contradicts the condition $U \cap A \notin I$. Hence, the assumption $x = a$ is false, which implies $x \neq a$, thus $x \in A$. Therefore, which gives α-closed, thus {a} is α-open, and consequently τ_α^* is the discrete topology.

Definition 3.2 An ideal topological space (X, τ, I) is called a α-T_0-space if for every two distinct points $x \neq y \in X$, there exists a α-open set containing exactly one of them and excluding the other.

Proposition 3.2 In an ideal topological space (X, τ, I), if for every pair of distinct points $x \neq y \, in \, X$ there exists an α-closed set containing exactly one of them and excluding the other, then the space is called an α-T_0- space.

Proof. Let x and y be two distinct points in X. By the definition of an α-T_0-space, there exists an α-open set U such that $x \in U$ but $y \notin U$. Looking at the complement of this set, we find that $X \setminus U$ is an α-closed set containing y but not x. Conversely, suppose there exists an α-closed set F such that $x \in F$ while $y \notin F$. Then its complement $X \setminus F$ is an α-open set containing y but not x, which is precisely what is required to satisfy the α-T_0 property of the space.

Definition 3.3 A space (X, τ_α^*) is said to be disconnected if there exist two non-empty α-open sets U and V such that $U \cap V = \emptyset$ and $U \cup V = X$.Equivalently, the space is disconnected if it contains a non-empty proper subset A that is both α-open and α-closed at the same time.

Theorem 3.4 The space (X, τ_α^*) is disconnected if and only if it contains a non-empty proper subset $A \subseteq X$ that is together α-open and α-closed.

Proof. ⇒ Suppose that (X, τ_α^*) is disconnected space, there exist two non-empty α-open sets U and such that $U \cap V = \emptyset$ and $U \cup V = X$. Since $V = X \setminus U$ and V is α-open, its complement U is α-closed. Thus, U is the non-empty proper subset that is both α-open and α-closed.

⇐ Conversely, if A is α-open and α-closed, there exists a non-empty proper subset A of X, Let $U = A$ and $V = X \setminus A$. We obtain two non-empty α-open sets satisfying $U \cap V = \emptyset$ and $U \cup V = X$, so the pair $\{U, V\}$ forms a separation of X.

Remark 3.1 To facilitate calculations and impart greater methodological clarity to computational operations, we will express the elements of Galois Field $GF(2^n)$ in numerical form rather than employing polynomial notation, while fully preserving the algebraic properties of the field structure.

4. Numerical Example in $GF(2^2)$:

To construct the field $GF(2^2)$, we utilize the irreducible polynomial $f(u) = u^2 + u + 1$ over Z_2. The elements of the field are the remainders of polynomials divided by $f(u)$, resulting in a set of $2^2 = 4$ distinct elements:

$$\{0,1,u,u+1\} \qquad (10)$$

This structure forms the finite field $GF(2^2)$, where addition in this field is defined by the usual polynomial addition modulo 2 and multiplication are performed modulo $f(u)$. Where we apply the u-adic valuation function $v_u(p)$ and forming the equivalence classes that serve as the topological basis for the space (X, R). This study begins with the power set of the finite field $X = GF(2^n)$, as it contains all possible subsets amounting to 2^n sets. From among these sets, those that satisfy the α-open condition are selected, and then the family upon which the α-local function is defined is formed when associated with the topological ideal I. Upon applying the closure operator to these sets, the α-ideal topology is constructed, which classifies the polynomials within $GF(2^n)$.

Example 4.1. Let $X = \{0,1,2,3\}$ be a finite fields , and define the equivalence relation R on X as follows: R = {(0,0) , (1,1) , (1,3) , (3,1) , (3,3) , (2,2)} ,The equivalence classes derived from this relation are $[u]_R = \{\{0\},\{1,3\},\{2\}\}$ and the B is a base of τ , $B = \{\{0\},\{1,3\},\{2\}\}$, $\tau = \{\varphi, X, \{0\}, \{1,3\}, \{2\}, \{0,1,3\}, \{0,2\}, \{1,2,3\}\}$

Case1. In this context, we define the ideal I on X in the following form I = $\{\emptyset,\{1\},\{3\},\{1,3\}\}$.

Table 1- The α-local function and α-I-closure, and α-open sets for all P(X)

P(X)	A^*_α	cl^*_α	α**-open**
$\emptyset$	$\emptyset$	$\emptyset$	Yes
{0}	{0}	{0}	Yes
{1}	$\emptyset$	{1}	No
{2}	{2}	{2}	Yes
{3}	$\emptyset$	{3}	No
{0,1}	{0}	{0,1}	No
{0,2}	{0,2}	{0,2}	Yes
{0,3}	{0}	{0,3}	No
{1,2}	{2}	{1,2}	No
{1,3}	$\emptyset$	{1,3}	Yes
{2,3}	{2}	{2,3}	No
{0,1,2}	{0,2}	{0,1,2}	No
{0,1,3}	{0}	{0,1,3}	Yes
{0,2,3}	{0,2}	{0,2,3}	No
{1,2,3}	{2}	{1,2,3}	Yes
X	{0,2}	X	Yes

Based on the calculations of the α-open sets, α-local function and cl^*_α, as shown in **Table 1**, the resulting topology is $\tau^*_\alpha =$ $\{\Phi,\{0\},\{1\},\{2\},\{3\},\{0,1\},\{0,2\},\{0,3\},\{1,2\},\{1,3\},\{2,3\},\{0,1,2\},\{0,1,3\},\{0,2,3\},\{1,2,3\},X\}$

Case2. In this context, we define the ideal I on X in the following form $I = \{\emptyset,\{1\},\{2\},\{1,2\}\}$.

Table 2- The α-local function and α∗-closure and α-open for all P(X)

P(X)	A_α^*	cl_α^***(A)**	α**-open**
∅	∅	∅	Yes
{0}	{0}	{0}	Yes
{1}	∅	{1}	No
{2}	∅	{2}	Yes
{3}	{1,3}	{1,3}	No
{0,1}	{0}	{0,1}	No
{0,2}	{0}	{0,2}	Yes
{0,3}	{0,1,3}	{0,1,3}	No
{1,2}	∅	{1,2}	No
{1,3}	{1,3}	{1,3}	Yes
{2,3}	{1,3}	{1,2,3}	No
{0,1,2}	{0}	{0,1,2}	No
{0,1,3}	{0,1,3}	{0,1,3}	Yes
{0,2,3}	{0,1,3}	X	No
{1,2,3}	{1,3}	{1,2,3}	Yes
X	{0,1,3}	X	Yes

Based on the calculations of the α-open sets, α-local function and cl_α^*, as shown in Table 2, the resulting topology is $\tau_\alpha^* = \{\emptyset, X, \{0\}, \{2\}, \{3\}, \{0,2\}, \{0,3\}, \{1,3\}, \{2,3\}, \{0,1,3\}, \{0,2,3\}, \{1,2,3\}\}$.

The results show that the choice of ideal significantly affects the induced topology. In the first case, using an ideal from the equivalence classes results in a discrete topology. However, in the second case, where the ideal is not from the equivalence classes, the induced space is a T_0-space.

5. Conclusions

The research revealed significant findings regarding the integration of algebraic structures with topological spaces through the mechanism of ideals, and the most important of these findings revolve around the following:

1. A direct link was established between polynomial rings and general topology, through the use of the valuation-based equivalence relation as a pivotal partitioning tool.

2. The research revealed that the α-local function effectively modifies topological structures, serving as a precise algebraic-topological tool that contributes to redefining set closures with high accuracy within ideal spaces.

3. The research proved that the topology τ_{α}^{*} is equivalent to the discrete topology $P(X)$ in finite spaces, provided that the ideal contains at least one complete equivalence class from the partition p.

4. The applied results clarified that the ideal I directly determines the nature of the resulting space, as the space (X,τ_{α}^{*}) transforms into a discrete and disconnected space simultaneously if the ideal belongs to the equivalence classes, and if it falls outside them, it transforms into a space of type α-T_0-space

References

[1] R. Lidl and H. Niederreiter, Finite Fields, 2nd ed. Cambridge: Cambridge University Press, 1997.

[2] J. B. Fraleigh, A First Course in Abstract Algebra, 7th ed. Pearson Education, Inc., 2003.

[3] G. T. Lee, Abstract Algebra: An Introductory Course, Springer International Publishing, Cham, Switzerland, 2018.

[4] R. Lidl and H. Niederreiter, Introduction to Finite Fields and Their Applications, Cambridge University Press, 1994.

[5] T. Mora, Solving Polynomial Equation Systems I: The Kronecker-Duval Philosophy, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 2003.

[6] D. Dikranjan and W. Tholen, Categorical Structure of Closure Operators, Mathematics and its Applications, Kluwer Academic Publishers, Dordrecht, Boston, 1995.

[7] Y. Jiang, A Practical Guide to Error-Control Coding Using MATLAB, Boston, MA, USA: Artech House, 2010.

[8] D. Andrijev´ıc, Some properties of the topology of α-sets, Mat. Vesnik., 16 (1984), 1–10.

[9] R. Vaidyanathaswamy, The Localization Theory in Set-Topology, *Proceedings of the Indian Academy of Sciences - Section A*, vol. 20, no. 1, pp. 51–61, 1945.

[10] K. Kuratowski, Topology, Vol. I, Academic Press, New York, 1966.

[11] A. A. Nasef, ”Ideals in General Topology” Ph.D. thesis, Tanta University, Egypt, 1992.

[12] K. Kuratowski, Topology I, Warszawa, 1933.

[13] M. E. Abd El-Monsef, E. F. Lashien and A. A. Nasef, On I-open sets and I-continuous functions, *Kyungpook Math. J.,* vol. 32, no. 1, pp. 21-30, 1992).

[14] M. Khan and T. Noiri, Semi-local functions in ideal topological spaces, *Journal of Advanced Research in Pure Mathematics*, vol. 2, no. 1, 36-42, 2010.

[15] M. E. Abd El-Monsef, A. A. Nasef, A. E. Radwan, and R. B. Esmaeel, ”On α-open sets with respect to an ideal,” *Journal of Advanced Studies in Topology*, vol. 5, no. 3, pp. 1–9, 2014. doi: 10.20454/jast.2014.781.

[16] A. A. Nasef, A. E. Radwan, and R. B. Esmaeel, “Some properties of α-open sets with respect to an ideal,” *International Journal of Pure and Applied Mathematics*, vol. 102, no. 4, pp. 613–630, 2015.

[17] M. E. Abd El-Monsef, A. A. Nasef, A. E. Radwan, and R. B. Esmaeel, “On α-open sets with respect to an ideal*, ” Journal of Advanced Studies in Topology*, vol. 5, no. 3, pp. 1–9, 2014.

[18] O. Nj˚astad, “On some classes of nearly open sets,” *Pacific Journal of Mathematics*, vol. 15, no. 3, pp. 961–970, 1965.

[19] F. Yalaz and A. K. Kaymakcı, "A new local function and a new compatibility type in ideal topological spaces," *AIMS Mathematics*, vol. 8, no. 3, pp. 7097–7114, 2023. doi: 10.3934/math.2023358.

[20] F. Yalaz and A. K. Kaymakcı, "New set types, decomposition of continuity and Γ-J-continuity via local closure function," October 2023.

[21] W. Al-Omeri and T. Noiri, "On semi-I-open sets, pre-I-open sets and e-I-open sets in ideal topological spaces," Published April 2021.