Sumer Journal for Pure Science (ISSN: 2790-7031)

SJPS 2025 (June), Vol. 4, No. 1, p.p. 203-222

Theoretical and Applied Analysis of Coincidence Fixed Points in Fuzzy Fréchet Space

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Abstract

This paper presents new results concerning coincidence and common fixed points in the setting of fuzzy Fréchet spaces, making essential use of the triangular inequality inherent in such spaces.to illustrate the applicability of the theoretical results, a concrete example is provided. Moreover, the established findings are employed to investigate a class of fuzzy differential equations, where we demonstrate the existence and uniqueness of a common fixed point for a triplet of operators. This fixed point corresponds to the unique solution of associated fuzzy differential system. The proposed approach opens new perspectives in the study of fuzzy dynamic systems and may serve as a foundation for extending similar results to broader classes of compatibility conditions and fuzzy differential models.

Keywords: The triangular inequality, Coincidence fixed point, fuzzy Fréchet spaces, and fuzzy semi-norm.

التحليل النظري والتطبيقي لنقاط التطابق الثابتة في فضاء فريشيه الضبابي

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الخلاصة

يقدم هذا البحث نتائج جديدة تتعلق بنقاط التطابق والنقاط الثابتة المشتركة ضمن فضاءات فريشيه الضبابية، مع الاستفادة من خاصية متباينة المثلث التي تميز هذه الفضاءات. ولتوضيح قابلية تطبيق النتائج النظرية، تم تقديم مثال ملموس. علاوةً على ذلك، تستخدم النتائج المثبتة لدراسة فئة من المعادلات التفاضلية الضبابية، حيث نثبت وجود وتفرد نقطة ثابتة مشتركة لثلاثية من العوامل. تقابل هذه النقطة الثابتة الحل الفريد للنظام التفاضلي الضبابي المرتبط بها. يفتح النهج المقترح افاقا جديدة في دراسة الانظمة الديناميكية الضبابية، وقد يشكل اساساً لتوسيع نطاق النتائج المُماثلة لتشمل فئات أوسع من شروط التوافق ونماذج التفاضل الضبابي.

1. Introduction

In recent years, the theory of fuzzy spaces has emerged as a fertile ground for advancing functional analysis within imprecise or uncertain frameworks. The foundation of this theory lies in the concept of fuzzy sets, first introduced by Lotfi A. Zadeh [1], which provided a mathematical framework to represent vagueness and ambiguity inherent in many real-world problems. Classical metric spaces, though powerful, often fall short when dealing with such uncertainty. This gap has been partially bridged by the development of fuzzy metric spaces, as initially introduced by Kramosil and Michalek [2], and further formalized by George and Veeramani [3]. Howerver, deeder generalizations that extend beyond fuzzy metric structures particularly toward topological vector space theory remain comparatively underexplored.

Motivated by the need for such generalizations, our work investigates a new direction: the incorporation of fuzzy norms into the framework of Fréchet spaces, resulting in what we term a fuzzy Fréchet space [4], [5]. Unlike fuzzy metric spaces, which focus primarily on pairwise distances under fuzziness, our approach explores the topological and linear structure of infinite-dimensional spaces in a fuzzy environment. This enables us to blend the rich topology of classical Fréchet spaces with the continuity, and fixed point behavior in non-crisp settings (see [6], [7], [8], [9], [10], [11]).

Building upon this foundation, the present work a new class of rational-type fuzzy contractions characterized by weak compatibility among three self-mappings within the framework of fuzzy Fréchet spaces. By utilizing the intrinsic triangular property of these spaces, we establish key results concerning coincidence and common fixed points, supported by relevant example. Moreover, we illustrate the applicability of the proposed framework by addressing a system of fuzzy differential equations, thereby demonstrating both the theoretical depth and practical utility of our approach. This contribution is poised to advance the understanding of fuzzy dynamics and stimulate further investigations across broader analytical settings.

2. Preliminaries

This section is devoted to presenting the fundamental notions, definitions, and results that will be used throughout the paper. For the reader's convenience, we include both standard concepts and specific tools relevant to our main discussion.

Definition 2.1 [12]. Consider two self-mappings T_1 and T_2 on a nonempty set E such that $T_1, T_2 \colon E \to E$. An element $t \in E$ is said to be a coincidence point of T_1 and T_2 if both mappings yield the same image at t, i.e., $T_1(t) = T_2(t)$. In such a case, the common image $y = T_1(t) = T_2(t)$ is referred to as a point of coincidence. The mappings T_1 and T_2 are described as weakly compatible provided they commute at their coincidence point; that is, whenever $T_1(t) = T_2(t)$, then $T_1\big(T_2(t)\big) = T_2\big(T_1(t)\big)$ holds.

Definition 2.2 [13]. A continuous t-norm is a binary operation $\#: [0,1] \times [0,1] \rightarrow [0,1]$ such that

i. # is associative, commutative, and continuous.

ii. $v_1\#1 = v_1$ and $v_1\#v_2 \leq v_3\#v_4$, $\forall v_1 \leq v_3$, and $v_2 \leq v_3$, where $v_{1,}v_2, v_3$, and $v_4 \in [0,1]$.

Definition 2.3 [14]. Let E be a vector space that spans a field K. A fuzzy set η in $E \times \mathbb{R}$ is referred to as a fuzzy semi-norm on E if the following conditions are satisfied:

i. $\eta(r, v_1) = 0, \forall v_1 \leq 0,$

ii. $\eta(\varrho r, v_1) = \eta\left(r, \frac{v_1}{\varrho}\right), \forall \varrho \in {}^{K}\!/_{\{0\}}\ , \forall v_1 > 0,$

iii. $\eta(r, v_1)\#\eta(s, v_2) \leq \eta(r + s, v_1 + v_2),$

iv. $\forall r \in E, \eta(r, v_1)$ is non-decreasing w.r.t v_1, $\lim_{v_1 \to \infty} \eta(r, v_1) = 1$, and $\lim_{v_1 \to 0} \eta(r, v_1) = 0.$

Definition 2.4 [14]. For each $r \neq 0$ in a vector space E, the family $G = \{\eta_i\}_{i \in I}$ of fuzzy semi-norms is said to be separating if at least one $\eta \in G$ and $v_1 > 0$ such that $\eta(r, v_1) \neq 1$ exists.

Definition 2.5 [4]. Suppose a vector space E is a complete fuzzy topological vector space, and its fuzzy topology τ_G is produced by a countable separating family of fuzzy semi-norms $G = \{\eta_i\}_{i \in I}$. In this case, it is called a fuzzy Fréchet space $(FF - Space)$.

In [4], the notions of fuzzy convergence, fuzzy Cauchy sequence, and fuzzy continuity in $FF - Space$ are discussed together with the building of fuzzy Fréchet space.

3. Main results

This section presents the main theorem and result concerning coincidence and common fixed points for three self-mappings in $F -$ spaces , established under rational-type fuzzy contractive conditions with weak compatibility.

Throughout this section, we assume that E is $FF -$ space and that the fuzzy semi-norm family $G = \{\eta_i\}_{j \in J}$ yields the fuzzy topology of E, and # is a continuous t-norm defined as $v_1\#v_2 = v_1.v_2, \forall v_1, v_2 \in [0,1]$.

Definition 2.6. Let $G = \{\eta_i\}_{i \in I}$ is a collection of fuzzy semi-norms in $FF - Space\ E$ that is triangular if

$$\left(\frac{1}{\eta_i(r-s,v)}-1\right) \leq \left(\frac{1}{\eta_i(r-u,v)}-1\right)+\left(\frac{1}{\eta_i(u-s,v)}-1\right)$$

$\forall r,s,u \in E, v>0$, and $\forall \eta_i \in G$.

Theorem 3.2. Let $T_1,T_2,T_3: E \to E$ be three self-mappings, that satisfy $\forall\, r,s \in E$,

$$\begin{aligned}&\frac{1}{\eta_i(T_1(r)-T_2(s),v)}-1\\ &\leq c_1\left(\frac{1}{\eta_i(T_3(r)-T_3(s),v)}-1\right)+c_2\left(\frac{1}{Q(\,T_1,T_2,T_3,r,s,v)}-1\right)\\ &+c_3\left(\frac{\eta_i(T_3(r)-T_1(r),v).\,\eta_i(T_3(s)-T_2(s),v)}{\eta_i(T_3(r)-T_2(s),2v).\,\eta_i(T_3(r)-T_3(s),v).\,\eta_i(T_3(s)-T_1(r),2v)}-1\right),\end{aligned} \tag{1}$$

Where

$$\begin{aligned}Q(\,T_1,T_2,T_3,r,s,v)\\ &=\max\{\eta_i(T_3(r)-T_3(s),v),\eta_i(T_3(r)-T_1(r),v),\eta_i(T_3(s)\\ &-T_2(s),v),\eta_i(T_3(s)-T_1(r),v),\eta_i(T_3(r)-T_2(s),v)\},\end{aligned} \tag{2}$$

For $(c_1+c_2+c_3)<1$ with $0 \leq c_1,c_2,c_3 <1$, $v>0$ and $\forall \eta_i \in G$, and let $G=\{\eta_i\}_{j\in J}$ is triangular in E. If $T_3(E)$ is a complete subspace of E, and $T_1(E) \cup T_2(E) \subset T_3(E)$. Then, there is point of coincidence in E for T_1, T_2 and T_3.

Proof: Assume that r_0 is an arbitrary point in E. Select a sequence $\{r_j\}$ in E using condition $T_1(E) \cup T_2(E) \subset T_3(E)$ so that

$$T_3(r_{2j+1})=T_1(r_{2j}) \text{ and } T_3(r_{2j+2})=T_2(r_{2j+1}), \forall j \geq 0. \tag{3}$$

By (1), for $v>0$ and $\forall \eta_i \in G$,, now

$$\frac{1}{\eta_i(T_3(r_{2j+1})-T_3(r_{2j+2}),v)}-1=\frac{1}{\eta_i(T_1(r_{2j})-T_2(r_{2j+1}),v)}-1$$
$$\leq c_1\left(\frac{1}{\eta_i\big(T_3(r_{2j})-T_3(r_{2j+1}),v\big)}-1\right)+c_2\left(\frac{1}{Q(\,T_1,T_2,T_3,r_{2j},r_{2j+1},v)}-1\right)$$
$$+c_3\left(\frac{\eta_i\big(T_3(r_{2j})-T_1(r_{2j}),v\big).\,\eta_i\big(T_3(r_{2j+1})-T_2(r_{2j+1}),v\big)}{\eta_i\big(T_3(r_{2j})-T_2(r_{2j+1}),2v\big).\,\eta_i\big(T_3(r_{2j})-T_3(r_{2j+1}),v\big).\,\eta_i\big(T_3(r_{2j+1})-T_1(r_{2j}),2v\big)}-1\right)$$
$$=c_1\left(\frac{1}{\eta_i\big(T_3(r_{2j})-T_3(r_{2j+1}),v\big)}-1\right)+c_2\left(\frac{1}{Q(\,T_1,T_2,T_3,r_{2j},r_{2j+1},v)}-1\right)$$
$$+c_3\left(\frac{\eta_i\big(T_3(r_{2j})-T_3(r_{2j+1}),v\big).\,\eta_i\big(T_3(r_{2j+1})-T_3(r_{2j+2}),v\big)}{\eta_i\big(T_3(r_{2j})-T_3(r_{2j+2}),2v\big).\,\eta_i\big(T_3(r_{2j})-T_3(r_{2j+1}),v\big)}-1\right)$$

(4)

Where

$$\begin{aligned}Q(\,T_1,T_2,T_3,r_{2j},r_{2j+1},v)&=\max\{\eta_i\big(T_3(r_{2j})-T_3(r_{2j+1}),v\big),\eta_i\big(T_3(r_{2j})-T_1(r_{2j}),v\big),\eta_i\big(T_3(r_{2j+1})-T_2(r_{2j+1}),v\big),\eta_i\big(T_3(r_{2j+1})-T_1(r_{2j}),v\big),\eta_i\big(T_3(r_{2j})-T_2(r_{2j+1}),v\big)\}\\&=\max\{\eta_i\big(T_3(r_{2j})-T_3(r_{2j+1}),v\big),\eta_i\big(T_3(r_{2j})-T_3(r_{2j+1}),v\big),\eta_i\big(T_3(r_{2j+1})-T_3(r_{2j+2}),v\big),\eta_i\big(T_3(r_{2j+1})-T_3(r_{2j+1}),v\big),\eta_i\big(T_3(r_{2j})-T_3(r_{2j+2}),v\big)\}\\&=\max\{\eta_i\big(T_3(r_{2j})-T_3(r_{2j+1}),v\big),\eta_i\big(T_3(r_{2j+1})-T_3(r_{2j+2}),v\big),1,\eta_i\big(T_3(r_{2j})-T_3(r_{2j+2}),v\big)\}=1.\end{aligned}$$

(5)

Now, using definition 2.3 (iii) and (4),(5), for $v>0$, we get

$$\frac{1}{\eta_i(T_3(r_{2j+1})-T_3(r_{2j+2}),v)}-1$$
$$\leq c_1\left(\frac{1}{\eta_i\big(T_3(r_{2j})-T_3(r_{2j+1}),v\big)}-1\right)$$
$$+c_3\left(\frac{\eta_i\big(T_3(r_{2j})-T_3(r_{2j+1}),v\big).\,\eta_i\big(T_3(r_{2j+1})-T_3(r_{2j+2}),v\big)}{\eta_i\big(T_3(r_{2j})-T_3(r_{2j+1}),v\big).\,\eta_i\big(T_3(r_{2j+1})-T_3(r_{2j+2}),v\big).\,\eta_i\big(T_3(r_{2j})-T_3(r_{2j+1}),v\big)}-1\right).$$

(6)

Once simplified, for $v>0$,

$$\frac{1}{\eta_i(T_3(r_{2j+1})-T_3(r_{2j+2}),v)}-1 \le \sigma\left(\frac{1}{\eta_i(T_3(r_{2j+1})-T_3(r_{2j+2}),v)}-1\right),$$

where $\sigma=(c_1+c_3)<1$.

(7)

Likewise, in light of (1), for $v>0$,

$$\frac{1}{\eta_i(T_3(r_{2j+2})-T_3(r_{2j+3}),v)}-1=\frac{1}{\eta_i(T_1(r_{2j+1})-T_2(r_{2j+2}),v)}-1$$
$$\le c_1\left(\frac{1}{\eta_i\big(T_3(r_{2j+1})-T_3(r_{2j+2}),v\big)}-1\right)+c_2\left(\frac{1}{Q(\,T_1,T_2,T_3,r_{2j+1},r_{2j+2},v)}-1\right)$$
$$+c_3\left(\frac{\eta_i\big(T_3(r_{2j+1})-T_1(r_{2j+1}),v\big).\eta_i\big(T_3(r_{2j+2})-T_2(r_{2j+2}),v\big)}{\eta_i\big(T_3(r_{2j+1})-T_2(r_{2j+2}),2v\big).\eta_i\big(T_3(r_{2j+1})-T_3(r_{2j+2}),v\big).\eta_i\big(T_3(r_{2j+2})-T_1(r_{2j+1}),2v\big)}\right.$$
$$\left.-1\right)$$
$$=c_1\left(\frac{1}{\eta_i\big(T_3(r_{2j+1})-T_3(r_{2j+2}),v\big)}-1\right)+c_2\left(\frac{1}{Q(\,T_1,T_2,T_3,r_{2j+1},r_{2j+2},v)}-1\right)$$
$$+c_3\left(\frac{\eta_i\big(T_3(r_{2j+1})-T_3(r_{2j+2}),v\big).\eta_i\big(T_3(r_{2j+2})-T_3(r_{2j+3}),v\big)}{\eta_i\big(T_3(r_{2j+1})-T_3(r_{2j+2}),2v\big).\eta_i\big(T_3(r_{2j+1})-T_3(r_{2j+2}),v\big)}-1\right)$$

(8)

Where

$$\begin{aligned}Q(\,T_1,T_2,T_3,r_{2j+1},r_{2j+2},v)&\\ =\max\{&\eta_i\big(T_3(r_{2j+1})-T_3(r_{2j+2}),v\big),\eta_i\big(T_3(r_{2j+1})\\ &-T_1(r_{2j+1}),v\big),\eta_i\big(T_3(r_{2j+2})-T_2(r_{2j+2}),v\big),\eta_i\big(T_3(r_{2j+2})\\ &-T_1(r_{2j+1}),v\big),\eta_i\big(T_3(r_{2j+1})-T_2(r_{2j+2}),v\big)\}\\ =\max\{&\eta_i\big(T_3(r_{2j+1})-T_3(r_{2j+2}),v\big),\eta_i\big(T_3(r_{2j+1})\\ &-T_3(r_{2j+2}),v\big),\eta_i\big(T_3(r_{2j+2})-T_3(r_{2j+3}),v\big),\eta_i\big(T_3(r_{2j+2})\\ &-T_3(r_{2j+2}),v\big),\eta_i\big(T_3(r_{2j+1})-T_3(r_{2j+3}),v\big)\}\\ =\max\{&\eta_i\big(T_3(r_{2j+1})-T_3(r_{2j+2}),v\big),\eta_i\big(T_3(r_{2j+2})\\ &-T_3(r_{2j+3}),v\big),1,\eta_i\big(T_3(r_{2j+1})-T_3(r_{2j+3}),v\big)\}=1.\end{aligned}$$

(9)

Now, using definition 2.3 (iii) and (8), (9), for $v>0$, we get

$$\frac{1}{\eta_i(T_3(r_{2j+2})-T_3(r_{2j+3}),v)}-1$$
$$\leq c_1\left(\frac{1}{\eta_i\big(T_3(r_{2j+1})-T_3(r_{2j+2}),v\big)}-1\right)$$
$$+\,c_3\left(\frac{\eta_i\big(T_3(r_{2j+1})-T_3(r_{2j+2}),v\big).\eta_i\big(T_3(r_{2j+2})-T_3(r_{2j+3}),v\big)}{\eta_i\big(T_3(r_{2j+1})-T_3(r_{2j+2}),v\big).\eta_i\big(T_3(r_{2j+2})-T_3(r_{2j+3}),v\big).\eta_i\big(T_3(r_{2j+1})-T_3(r_{2j+2}),v\big)}\right.$$
$$\left.-1\right).$$

(10)

Then Once simplified, for $v > 0$,

$$\frac{1}{\eta_i(T_3(r_{2j+2})-T_3(r_{2j+3}),v)}-1 \leq \sigma(\frac{1}{\eta_i(T_3(r_{2j+2})-T_3(r_{2j+3}),v)}-1),$$

where $\sigma = (c_1 + c_3) < 1$.

(11)

Currently, based on (7), (11), and induction,

$$\frac{1}{\eta_i(T_3(r_{2j+2})-T_3(r_{2j+3}),v)}-1 \leq \sigma\left(\frac{1}{\eta_i(T_3(r_{2j+1})-T_3(r_{2j+2}),v)}-1\right)$$
$$\leq \sigma^2\left(\frac{1}{\eta_i(T_3(r_{2j})-T_3(r_{2j+1}),v)}-1\right) \leq \cdots$$
$$\leq \sigma^{2j+2}\left(\frac{1}{\eta_i(T_3(r_0)-T_3(r_1),v)}-1\right) \to 0, as\ j \to \infty.$$

(12)

As a result, $\{T_3(r_j)\}$ is fuzzy contractive sequence in FF −space E,

$$\lim_{j\to\infty} \eta_i(T_3(r_j)-T_3(r_{j-1}),v) = 1 \text{ for } v > 0 \qquad (13)$$

Given that $G = \{\eta_i\}_{j\in J}$is triangular. $k > j > j_0$,

$$\frac{1}{\eta_i(T_3(r_j)-T_3(r_k),v)}-1$$
$$\leq\left(\frac{1}{\eta_i(T_3(r_j)-T_3(r_{j+1}),v)}-1\right)+\left(\frac{1}{\eta_i(T_3(r_{j+1})-T_3(r_{j+2}),v)}-1\right)$$
$$+\cdots+\left(\frac{1}{\eta_i(T_3(r_{j-1})-T_3(r_j),v)}-1\right)$$
$$\leq\sigma^j\left(\frac{1}{\eta_i(T_3(r_j)-T_3(r_{j+1}),v)}-1\right)$$
$$+\sigma^{j+1}\left(\frac{1}{\eta_i(T_3(r_{j+1})-T_3(r_{j+2}),v)}-1\right)+\cdots$$
$$+\sigma^{j-1}\left(\frac{1}{\eta_i(T_3(r_{j-1})-T_3(r_j),v)}-1\right)$$
$$\leq\left(\sigma^j+\sigma^{j+1}+\cdots+\sigma^{j-1}\right)\left(\frac{1}{\eta_i(T_3(r_0)-T_3(r_1),v)}-1\right)$$
$$\leq\left(\frac{\sigma^j}{1-\sigma}\right)\left(\frac{1}{\eta_i(T_3(r_0)-T_3(r_1),v)}-1\right)\to 0, as\ j\to\infty.$$

(14)

This demonstrates that the sequence $\{T_3(r_j)\}$ is fuzzy Fréchet-Cauchy sequence in FF −space E and since $T_3(E)$ is a complete subspace of E. Thus, there are y and t in E such that $T_3(r_j)\to y=T_3(t)\ as\ j\to\infty$, i.e.

$$\lim_{j\to\infty}\eta_i(y-T_3(r_j),v)=\eta_i(y-T_3(t),v)=1, \text{ for } v>0.$$

(15)

Given that $G=\{\eta_i\}_{j\in J}$is triangular,

$$\frac{1}{\eta_i(T_3(t)-T_1(t),v)}-1$$
$$\leq\left(\frac{1}{\eta_i(T_3(t)-T_3(r_{2j+2}),v)}-1\right)+\left(\frac{1}{\eta_i(T_3(r_{2j+2})-T_1(t),v)}-1\right),$$

for $v>0$.

(16)

Now, using definition 2.3 (iii) and (1), (13), (15), for $v>0$, we get

$$\frac{1}{\eta_i(T_3(r_{2j+2})-T_1(t),v)}-1=\frac{1}{\eta_i(T_1(t)-T_2(r_{2j+1}),v)}-1\leq c_1\left(\frac{1}{\eta_i(T_3(t)-T_3(r_{2j+1}),v)}-1\right)+$$
$$c_2\left(\frac{1}{Q(T_1,T_2,T_3,t,r_{2j+1},v)}-1\right)+$$
$$c_3\left(\frac{\eta_i(T_3(t)-T_1(t),v).\eta_i(T_3(r_{2j+1})-T_2(r_{2j+1}),v)}{\eta_i(T_3(t)-T_2(r_{2j+1}),2v).\eta_i(T_3(t)-T_3(r_{2j+1}),v).\eta_i(T_3(r_{2j+1})-T_1(t),2v)}-1\right)\leq$$

$$c_1\left(\frac{1}{\eta_i(T_3(t)-T_3(r_{2j+1}),v)}-1\right)+c_2\left(\frac{1}{Q(T_1,T_2,T_3,t,r_{2j+1},v)}-1\right)+$$
$$c_3\left(\frac{\eta_i(T_3(t)-T_1(t),v).\eta_i(T_3(r_{2j+1})-T_3(r_{2j+2}),v)}{\eta_i(T_3(t)-T_3(r_{2j+2}),2v).\eta_i(T_3(t)-T_3(r_{2j+1}),v).\eta_i(T_3(r_{2j+1})-T_3(t),v).\eta_i(T_3(t)-T_1(t),v)}-1\right)\to$$
$$c_2\left(\frac{1}{Q(T_1,T_2,T_3,t,r_{2j+1},v)}-1\right)\text{as } j\to\infty.$$

(17)

Where

$$\begin{aligned}Q(T_1,T_2,T_3,t,r_{2j+1},v)&=\max\{\eta_i(T_3(t)-T_3(r_{2j+1}),v),\eta_i(T_3(t)-T_1(t),v),\eta_i(T_3(r_{2j+1})\\&-T_2(r_{2j+1}),v),\eta_i(T_3(r_{2j+1})-T_1(t),v),\eta_i(T_3(t)-T_2(r_{2j+1}),v)\}\\&=\max\{\eta_i(T_3(t)-T_3(r_{2j+1}),v),\eta_i(T_3(t)-T_1(t),v),\eta_i(T_3(r_{2j+1})\\&-T_3(r_{2j+2}),v),\eta_i(T_3(r_{2j+1})-T_1(t),v),\eta_i(T_1(t)-T_3(r_{2j+2}),v)\}\\&\to\max\{1,\eta_i(T_3(t)-T_1(t),v)\}=1, as\ j\to\infty\end{aligned}$$

(18)

Now, using (17) and (18), for $v>0$, we get

$$\limsup_{j\to\infty}\left(\frac{1}{\eta_i(T_3(r_{2j+2})-T_1(t),v)}-1\right)=0, v>0.$$

(19)

We obtain that $\eta_i(T_3(t)-T_1(t),v)=1\Rightarrow y=T_3(t)=T_1(t)$ for $v>0$ by using (15) and (19) in (16) with $j\to\infty$. The next step is to demonstrate that $y=T_3(t)=T_1(t)$. Given that $G=\{\eta_i\}_{j\in J}$ is triangular, then

$$\frac{1}{\eta_i(T_3(t)-T_2(t),v)}-1\le\left(\frac{1}{\eta_i(T_3(t)-T_3(r_{2j+1}),v)}-1\right)+\left(\frac{1}{\eta_i(T_3(r_{2j+1})-T_2(t),v)}-1\right),$$

for $v>0$.

(20)

Now, using definition 2.3 (iii) and (1), (13), (15), for $v>0$, we get

$$\frac{1}{\eta_i(T_3(r_{2j+1})-T_2(t),v)}-1=\frac{1}{\eta_i(T_1(r_{2j})-T_2(t),v)}-1$$
$$\le c_1\left(\frac{1}{\eta_i\big(T_3(r_{2j})-T_3(t),v\big)}-1\right)+c_2\left(\frac{1}{Q(\,T_1,T_2,T_3,t,r_{2j},v)}-1\right)$$
$$+c_3\left(\frac{\eta_i\big(T_3(r_{2j})-T_1(r_{2j}),v\big).\eta_i(T_3(t)-T_2(t),v)}{\eta_i\big(T_3(r_{2j})-T_2(t),2v\big).\eta_i\big(T_3(r_{2j})-T_3(r_{2j}),v\big).\eta_i\big(T_3(t)-T_1(r_{2j}),2v\big)}\right.$$
$$\left.-1\right)$$
$$\le c_1\left(\frac{1}{\eta_i\big(T_3(r_{2j})-T_3(t),v\big)}-1\right)+c_2\left(\frac{1}{Q(\,T_1,T_2,T_3,t,r_{2j},v)}-1\right)$$
$$+c_3\left(\frac{\eta_i\big(T_3(r_{2j})-T_3(r_{2j+1}),v\big).\eta_i(T_3(t)-T_2(t),v)}{\eta_i\big(T_3(r_{2j})-T_3(t),2v\big).\eta_i(T_3(t)-T_2(t),v).\eta_i\big(T_3(r_{2j})-T_3(t),v\big).\eta_i\big(T_3(t)-T_3(r_{2j+1}),v\big)}\right.$$
$$\left.-1\right)\to c_2\left(\frac{1}{Q(\,T_1,T_2,T_3,t,r_{2j},v)}-1\right)\text{as } j\to\infty.$$

(21)

Where

$$\begin{aligned}Q(\,T_1,T_2,T_3,t,r_{2j},v)&\\ &=\max\{\eta_i\big(T_3(r_{2j})-T_3(t),v\big),\eta_i\big(T_3(r_{2j})-T_1(r_{2j}),v\big),\eta_i(T_3(t)\\ &-T_2(t),v),\eta_i\big(T_3(t)-T_1(r_{2j}),v\big),\eta_i\big(T_3(r_{2j})-T_2(t),v\big)\}\\ &=\max\{\eta_i\big(T_3(r_{2j})-T_3(t),v\big),\eta_i\big(T_3(r_{2j})-T_3(r_{2j+1}),v\big),\eta_i(T_3(t)\\ &-T_2(t),v),\eta_i\big(T_3(t)-T_3(r_{2j+1}),v\big),\eta_i\big(T_3(r_{2j})-T_2(t),v\big)\}\\ &\to\max\{1,\eta_i(T_3(t)-T_2(t),v)\}\ =1\end{aligned}$$

as $j\to\infty$.

(22)

Now, using (21) and (22), for $v>0$, we get

$$\lim_{j\to\infty}\sup\left(\frac{1}{\eta_i(T_3(r_{2j+1})-T_2(t),v)}-1\right)=0,v>0.$$

(23)

We obtain that $\eta_i(T_3(t)-T_2(t),v)=1\Rightarrow y=T_3(t)=T_2(t)$ for $v>0$ by using (15) and (23) in (20) with $j\to\infty$. With $y=T_3(t)=T_2(t)=T_1(t)$, we can conclude that y is a point of coincidence of the mappings T_1,T_2, and T_3 in E. ■

Results 3.3. Let $T_1,T_2,T_3\colon E\to E$ be three self-mappings, that satisfy $\forall\, r,s\in E$,

$$\frac{1}{\eta_i(T_1(r)-T_2(s),v)}-1 \le c_1\left(\frac{1}{\eta_i(T_3(r)-T_3(s),v)}-1\right)+c_2\left(\frac{1}{Q(\,T_1,T_2,T_3,r,s,v)}-1\right), \tag{24}$$

Where

$$Q(\,T_1,T_2,T_3,r,s,v) = \max\{\eta_i(T_3(r)-T_3(s),v),\eta_i(T_3(r)-T_1(r),v),\eta_i(T_3(s)-T_2(s),v),\eta_i(T_3(s)-T_1(r),v),\eta_i(T_3(r)-T_2(s),v)\}, \tag{25}$$

For $(c_1+c_2)<1$ with $0\le c_1,c_2 <1$, $v>0$ and $\forall\eta_i \in G$, and let $G=\{\eta_i\}_{j\in J}$ is triangular in E. If $T_3(E)$ is a complete subspace of E, and $T_1(E)\cup\, T_2(E)\subset T_3(E)$. Then, there is a point of coincidence in E for T_1, T_2, and T_3. Furthermore, if (T_1, T_3) and (T_2, T_3) are weakly compatible pairings, then each of T_1, T_2, and T_3 has a single common fixed point in E.

Proof. According to theorem (3.2)'s proof , y is a point of coincidence of the T_1, T_2, and T_3 in E. We now demonstrate that y is unique, let $y_1=T_1(t_1)=T_2(t_1)=T_3(t_1)$ for some $t_1\in E$, where y_1 is another point of coincidence of the mappings T_1, T_2, and T_3 in E. Then, for $v>0$, from (24),

$$\frac{1}{\eta_i(y-y_1,v)}-1=\frac{1}{\eta_i(T_3(t)-T_3(t_1),v)}-1=\frac{1}{\eta_i(T_1(t)-T_2(t_1),v)}-1 \le c_1\left(\frac{1}{\eta_i(T_3(t)-T_3(t_1),v)}-1\right)+c_2\left(\frac{1}{Q(\,T_1,T_2,T_3,t,t_1,v)}-1\right), \tag{26}$$

Where

$$\begin{aligned}Q(\,T_1,T_2,T_3,t,t_1,v) &= \max\{\eta_i(T_3(t)-T_3(t_1),v),\eta_i(T_3(t)-T_1(t),v),\eta_i(T_3(t_1)-T_2(t_1),v),\eta_i(T_3(t_1)-T_1(t),v),\eta_i(T_3(t)-T_2(t_1),v)\}\\ &= \max\{\eta_i(T_3(t)-T_3(t_1),v),1\}=1.\end{aligned} \tag{27}$$

So,

$$\frac{1}{\eta_i(y-y_1,v)}-1\le c_1\left(\frac{1}{\eta_i(T_3(t)-T_3(t_1),v)}-1\right)=c_1\left(\frac{1}{\eta_i(t-t_1,v)}-1\right), \tag{28}$$

for $v > 0$ and $\forall \eta_i \in G$. Since $(1 - c_1) \neq 0$ for $(c_1 + c_2) < 1$, thus, $\eta_i(t - t_1, v) = 1 \Longrightarrow t = t_1$, for $v > 0$ and $\forall \eta_i \in G$. A single common fixed point of the mappings $T_1, T_2,$ and T_3 can be obtained by applying Proportion(1.4) in [10] and the weak compatibility of the pair $(T_1, T_3), (T_2, T_3)$. Assuming there is $m \in E$ such that $m = T_3(m) = T_2(m) = T_1(m)$. Therefore, for $v > 0$ and $\forall \eta_i \in G$, we obtain that $\eta_i(t - m, v) = 1 \Longrightarrow t = m$. ■

Example 3.4. Let $E = [0, 1]$ be a FF −space with fuzzy semi-norm $\eta: E \times (0, \infty) \to [0,1]$ has been defined by

$$\eta(r, v) = \frac{v}{v + |r|}, \forall r \in E, and\ v > 0.$$

Then, demonstrating that η is triangular is simple. Let's define the mappings $T_1, T_2, T_3: E \to E$ as

$T_1(r) = T_2(r) = \frac{4r}{3r+9}$ and $T_3(r) = \frac{2r}{3}, \forall r \in E.$

And so we have

$$\begin{aligned} \frac{1}{\eta(T_1(r) - T_2(s), v)} - 1 &= \frac{1}{v}|T_1(r) - T_2(s)| = \frac{1}{v}\left|\frac{4r}{3r+9} - \frac{4s}{3s+9}\right| \\ &= \frac{1}{v}\left|\frac{36r - 36s}{(3r+9)(3s+9)}\right| \leq \frac{1}{v}\left|\frac{36r - 36s}{81}\right| = \frac{2}{3}\left|\frac{T_3(r) - T_3(s)}{v}\right| \\ &= \frac{2}{3}\left(\frac{1}{\eta(T_3(r) - T_3(s), v)} - 1\right), \text{for } v > 0. \end{aligned}$$

Therefore, in FF −space E, the weakly compatible fuzzy contractive requirement is satisfied by $T_1, T_2, T_3: E \to E$, that is,

$$\frac{1}{\eta(T_1(r) - T_2(s), v)} - 1 = \frac{2}{3}\left(\frac{1}{\eta(T_3(r) - T_3(s), v)} - 1\right), \text{ for } v > 0. \tag{29}$$

The value of the second term, which appears in (24), is then determined. Next, we have the cases listed below:

1. If for $t > 0$, the maximum value of $Q(T_1, T_2, T_3, r, s, v) = \eta(T_3(r) - T_3(s), v)$, for $v > 0$. Then,

$$\frac{1}{Q(T_1, T_2, T_3, r, s, v)} - 1 = \frac{1}{\eta(T_3(r) - T_3(s), v)} - 1 = \frac{1}{v}|T_3(r) - T_3(s)| = \frac{2}{3v}|r - s|, \text{ for } v > 0. \tag{30}$$

2. If for $t > 0$, the maximum value of $Q(T_1, T_2, T_3, r, s, v) = \eta(T_3(r) - T_1(r), v)$, for $v > 0$. Then,

$$\frac{1}{Q(T_1,T_2,T_3,r,s,v)}-1=\frac{1}{\eta(T_3(r)-T_1(r),v)}-1=\frac{1}{v}|T_3(r)-T_1(r)|=\frac{1}{v}\left|\frac{2r}{3}-\frac{4r}{3r+9}\right|=\frac{1}{v}\left|\frac{2(r^2+r)}{3r+9}\right|\le\frac{2}{9v}(r^2+r),\text{ for } v>0. \tag{31}$$

3. If for $t>0$, the maximum value of $Q(T_1,T_2,T_3,r,s,v)=\eta(T_3(s)-T_2(s),v)$, for $v>0$. Then,

$$\frac{1}{Q(T_1,T_2,T_3,r,s,v)}-1=\frac{1}{\eta(T_3(s)-T_2(s),v)}-1=\frac{1}{v}|T_3(s)-T_2(s)|=\frac{1}{v}\left|\frac{2s}{3}-\frac{4s}{3s+9}\right|=\frac{1}{v}\left|\frac{2(s^2+s)}{3s+9}\right|\le\frac{2}{9v}(s^2+s),\text{ for } v>0 \tag{32}$$

4. If for $t>0$, the maximum value of $Q(T_1,T_2,T_3,r,s,v)=\eta(T_3(s)-T_1(r),v)$, for $v>0$. Then,

$$\frac{1}{Q(T_1,T_2,T_3,r,s,v)}-1=\frac{1}{\eta(T_3(s)-T_1(r),v)}-1=\frac{1}{v}|T_3(s)-T_1(r)|=\frac{1}{v}\left|\frac{2s}{3}-\frac{4r}{3r+9}\right|=\frac{1}{v}\left|\frac{2(rs+3s-2r)}{3s+9}\right|\le\frac{2}{9v}|rs+3s-2r|,\text{ for } v>0 \tag{33}$$

5. If for $t>0$, the maximum value of $Q(T_1,T_2,T_3,r,s,v)=\eta(T_3(r)-T_2(s),v)$, for $v>0$. Then,

$$\frac{1}{Q(T_1,T_2,T_3,r,s,v)}-1=\frac{1}{\eta(T_3(r)-T_2(s),v)}-1=\frac{1}{v}|T_3(r)-T_2(s)|=\frac{1}{v}\left|\frac{2r}{3}-\frac{4s}{3s+9}\right|=\frac{1}{v}\left|\frac{2(rs+3r-2s)}{3r+9}\right|\le\frac{2}{9v}|rs+3r-2s|,\text{ for } v>0 \tag{34}$$

Therefore, when (29) is added to all the cases and contacts $c_1=\frac{2}{3}$ and $c_2=\frac{2}{7}$, we have

$$\frac{1}{\eta(T_1(r)-T_2(s),v)}-1\le\frac{2}{3}\left(\frac{1}{\eta(T_3(r)-T_3(s),v)}-1\right)+\frac{2}{7}\left(\frac{1}{Q(T_1,T_2,T_3,r,s,v)}-1\right), \tag{35}$$

Where

$$Q(T_1,T_2,T_3,r,s,v)=\max\{\eta(T_3(r)-T_3(s),v),\eta(T_3(r)-T_1(r),v),\eta(T_3(s)-T_2(s),v),\eta(T_3(s)-T_1(r),v),\eta(T_3(r)-T_2(s),v)\},\text{ for } v>0. \tag{36}$$

Therefore, for $c_1 = \frac{2}{3}$ and $c_2 = \frac{2}{7}$, all of hypotheses of result (3.3) are satisfied, and the mappings T_1, T_2 and T_3 have a single common fixed point, specifically 0.

4. Application

To support and demonstrate the key findings of our work, we provide an application of fuzzy differential equations $(FDEs)$ in this section.

Let H denote the set of all fuzzy subsets m defined on the real numbers $\mathbb{R}$. The $(FDEs)$ we have are as follows.

$$m''(t) = g\big(t, m(t), m'(t)\big), t \in [c_1, c_2],$$

$$m(t_1) = m_1, m(t_2) = m_2, t_1, t_2 \in [c_1, c_2], \tag{37}$$

Where the function $g: [c_1, c_2] \times H \times H \to H$ is continuous. The following integral equation is equivalent to this problem

$$m(t) = \int_{t_1}^{t_2} U(t, \vartheta)\left(g\big(\vartheta, m(\vartheta), m'(\vartheta)\big)\right) d\vartheta + \psi(t), \tag{38}$$

Green's function U is provided by

$$U(t, \vartheta) = \begin{cases} \dfrac{(t_2 - t)(\vartheta - t_1)}{t_2 - t_1}, t_1 \le \vartheta \le t \le t_2, \\ \dfrac{(t_2 - \vartheta)(t - t_1)}{t_2 - t_1}, t_1 \le t \le \vartheta \le t_2. \end{cases} \tag{39}$$

Additionally, $\psi(t)$ satisfies $\psi'' = 0, \psi(t_1) = m_1, \psi(t_2) = m_2$. In this case, we remember a few characteristics of U, including

$$\int_{t_1}^{t_2} U(t, \vartheta) d\vartheta \le \frac{(t_2 - t_1)^2}{8},$$

$$\int_{t_1}^{t_2} U_t(t, \vartheta) d\vartheta \le \frac{t_2 - t_1}{2}. \tag{40}$$

Let $E = C^1([c_1, c_2], H)$ be a FF −space with triangular fuzzy semi-norm $\eta: E \times (0, \infty) \to [0,1]$ has been defined by

$$\eta(r,v)=\frac{v}{v+|r|},\forall r\in E, and\ v>0, \quad (41)$$

Where $C^1([c_1,c_2],H)$ represent the set of all functions from $[c_1,c_2]$ to H that possess continuous first derivatives throughout the interval $[c_1,c_2]$.

We now use result(3.3) to demonstrate the current outcome for the boundary value problem mentioned above.

Theorem 4.1. Let $g_1,g_2:[c_1,c_2]\times H\times H\to H$ and let $\exists\delta,\gamma\in(0,1)$ with $\delta\le\gamma$ such that $\forall m,s\in E$, satisfies

$$\left|g_1\big(t,m(t),m'(t)\big)-g_2\big(t,s(t),s'(t)\big)\right|\le\delta|m(t)-s(t)|+\gamma|m'(t)-s'(t)|. \quad (42)$$

Let $\exists\theta$ such that $0<\theta<1$ and

$$|m(t)-s(t)|=|m-s|\le\theta\aleph(T_1,T_2,T_3,m,s), \quad (43)$$

Where

$$\aleph(T_1,T_2,T_3,m,s)=\max\{|T_3m-T_3s|,|T_3m-T_1m|,|T_3s-T_2s|,|T_3s-T_1m|,|T_3m-T_2s|\}. \quad (44)$$

Then in $E=C^1([t_1,t_2],H)$, the integral equations

$$m(t)=\int_{t_1}^{t_2}U(t,\vartheta)\left(g_1\big(\vartheta,m(\vartheta),m'(\vartheta)\big)\right)d\vartheta+\psi(t),t\in[c_1,c_2],$$

$$s(t)=\int_{t_1}^{t_2}U(t,\vartheta)\left(g_2\big(\vartheta,s(\vartheta),s'(\vartheta)\big)\right)d\vartheta+\psi(t),t\in[c_1,c_2], \quad (45)$$

admit a unique common solution.

Proof. Let $E=C^1([t_1,t_2],H)$ with semi-norm

$$P(m)=\max_{t_1\le t\le t_2}(\delta|m(t)|+\gamma|m'(t)|) \quad (46)$$

The space $E=C^1([t_1,t_2],H)$ is Fréchet space. At this point, we define the operators $T_1,T_2,T_3:E\to E$ as $T_1(m)=A_m+\psi,T_2(m)=B_m+\psi,T_3(m)=m,$ and $T_3(s)=s,$ (47)

Where

$$A_m = \int_{t_1}^{t_2} U(t,\vartheta)\left(g_1\big(\vartheta, m(\vartheta), m'(\vartheta)\big)\right) d\vartheta, t \in [c_1, c_2],$$

$$B_m = \int_{t_1}^{t_2} U(t,\vartheta)\left(g_2\big(\vartheta, s(\vartheta), s'(\vartheta)\big)\right) d\vartheta, t \in [c_1, c_2],$$

(48)

Where $g_1, g_2 \in C([c_1, c_2] \times H \times H, H), m, s \in C^1([c_1, c_2], H)$, and $\psi \in C([c_1, c_2], H)$.

Then by the characteristics of U, and from (46), (47) and applying the hypothesis, we get

$$|T_1 m(t) - T_2 s(t)| \leq \int_{t_1}^{t_2} |U(t,\vartheta)| \left| g_1\big(\vartheta, m(\vartheta), m'(\vartheta)\big) - g_2\big(\vartheta, s(\vartheta), s'(\vartheta)\big) \right| d\vartheta$$
$$\leq P(m-s) \int_{t_1}^{t_2} |U(t,\vartheta)| d\vartheta \leq \frac{(t_2 - t_1)^2}{8} P(m-s) \leq \frac{P(m-s)}{8},$$

$$|(T_1 m)'(t) - (T_2 s)'(t)|$$
$$\leq \int_{t_1}^{t_2} |U_t(t,\vartheta)| \left| g_1\big(\vartheta, m(\vartheta), m'(\vartheta)\big) - g_2\big(\vartheta, s(\vartheta), s'(\vartheta)\big) \right| d\vartheta$$
$$\leq P(m-s) \int_{t_1}^{t_2} |U_t(t,\vartheta)| d\vartheta \leq \frac{t_2 - t_1}{2} P(m-s) \leq \frac{P(m-s)}{2}.$$

(49)

Now, from (42), (46), and the above, we have that

$$P(T_1 m - T_2 s) = \max_{t_1 \leq t \leq t_2} (\delta |T_1 m(t) - T_2 s(t)| + \gamma |(T_1 m)'(t) - (T_2 s)'(t)|)$$
$$\leq \delta \frac{P(m-s)}{8} + \gamma \frac{P(m-s)}{2} \leq \left(\frac{5}{8}\gamma\right) P(m-s).$$

(50)

By (43), we now have that

$$P(T_1 m - T_2 s) \leq \left(\frac{5}{8}\gamma\right) P(m-s) \leq \varrho \aleph(T_1, T_2, T_3, m, s),$$

(51)

Where $\varrho = \frac{5}{8}\gamma\theta < 1$. We now apply result (3.3) to show that T_1, T_2 and T_3 admit a unique common fixed point $m_1 \in E$, i.e. $m_1 \in E$ is a BVP solution. There are the cases we have.

i. In (44), if

$$\aleph(T_1, T_2, T_3, m, s) = \max\{|T_3m - T_3s|, |T_3m - T_1m|, |T_3s - T_2s|, |T_3s - T_1m|, |T_3m - T_2s|\} = |T_3m - T_3s|.$$

Then from (41) and (51), we get

$$\frac{1}{\eta(T_1m - T_2s, v)} - 1 = \frac{P(T_1m - T_2s)}{v} \leq \frac{\varrho\aleph(T_1, T_2, T_3, m, s)}{v} = \frac{\varrho|T_3m - T_3s|}{v} = \varrho\left(\frac{1}{\eta(T_3m - T_3s, v)} - 1\right). \tag{52}$$

Thus $\frac{1}{\eta(T_1m-T_2s,v)} - 1 \leq \varrho\left(\frac{1}{\eta(T_3m-T_3s,v)} - 1\right)$, for $v > 0$, (53)

$\forall m, s \in E$. Accordingly, the operators T_1, T_2 and T_3 meet all the requirements of result (3.3) with $\varrho = (c_1 + c_2)$ as given in equation (24). As a result, T_1, T_2 and T_3 admit a unique common fixed point $m_1 \in E$, i.e. $m_1 \in E$ is a BVP solution.

ii. In (44), if

$$\aleph(T_1, T_2, T_3, m, s) = \max\{|T_3m - T_3s|, |T_3m - T_1m|, |T_3s - T_2s|, |T_3s - T_1m|, |T_3m - T_2s|\} = |T_3m - T_1m|.$$

Then from (41) and (51), we get

$$\frac{1}{\eta(T_1m - T_2s, v)} - 1 = \frac{P(T_1m - T_2s)}{v} \leq \frac{\varrho\aleph(T_1, T_2, T_3, m, s)}{v} = \frac{\varrho|T_3m - T_1m|}{v} = \varrho\left(\frac{1}{\eta(T_3m - T_1m, v)} - 1\right). \tag{54}$$

Thus $\frac{1}{\eta(T_1m-T_2s,v)} - 1 \leq \varrho\left(\frac{1}{\eta(T_3m-T_1m,v)} - 1\right)$, for $v > 0$, and $\forall m, s \in E$. (55)

In a similar manner, we arrive at the following three cases:

iii. In (44), if

$$\aleph(T_1, T_2, T_3, m, s) = \max\{|T_3m - T_3s|, |T_3m - T_1m|, |T_3s - T_2s|, |T_3s - T_1m|, |T_3m - T_2s|\} = |T_3s - T_2s|.$$

Then from (41) and (51), we get

$$\frac{1}{\eta(T_1m - T_2s, v)} - 1 \leq \varrho\left(\frac{1}{\eta(T_3s - T_2s, v)} - 1\right), \text{ for } v > 0, \text{ and } \forall m, s \in E. \quad (56)$$

iv. In (44), if

$$\aleph(T_1, T_2, T_3, m, s) = \max\{|T_3m - T_3s|, |T_3m - T_1m|, |T_3s - T_2s|, |T_3s - T_1m|, |T_3m - T_2s|\} = |T_3s - T_1m|.$$

Then from (41) and (51), we get

$$\frac{1}{\eta(T_1m - T_2s, v)} - 1 \leq \varrho\left(\frac{1}{\eta(T_3s - T_1m, v)} - 1\right), \text{ for } v > 0, \text{ and } \forall m, s \in E. \quad (57)$$

v. In (44), if

$$\aleph(T_1, T_2, T_3, m, s) = \max\{|T_3m - T_3s|, |T_3m - T_1m|, |T_3s - T_2s|, |T_3s - T_1m|, |T_3m - T_2s|\} = |T_3m - T_2s|.$$

Then from (41) and (51), we get

$$\frac{1}{\eta(T_1m - T_2s, v)} - 1 \leq \varrho\left(\frac{1}{\eta(T_3m - T_2s, v)} - 1\right), \text{ for } v > 0, \text{ and } \forall m, s \in E. \quad (58)$$

Therefore, based on (55), (56), (57), and (58), it is evident that the operators T_1, T_2 and T_3 fulfill all the assumptions outlined in result (3.3), taking $c_1 = 0$, and $\varrho = c_2$ as specified in (24). Consequently, there exists a unique common fixed point $m_1 \in E$ for these operators, which corresponds to a solution of the boundary value problem presented in (37).

5. Conclusion

In this work, we have introduced and explored a novel class of rational type weakly-compatible fuzzy contractions involving three self-mappings in fuzzy Fréchet spaces.

By leveraging the triangular property of fuzzy Fréchet spaces, we established several fundamental results concerning coincidence points and common fixed points, supported by an illustrative example. Furthermore, we demonstrated the practical relevance of our theoretical framework by applying it to a class of fuzzy differential equations, where we proved the existence and uniqueness of a common fixed point corresponding to the solution of the system.

This study not only enhances the understanding of fixed point theory in fuzzy settings but also opens up promising pathways for future research. The developed methodology can be extended to broader contexts involving different types of fuzzy contractions and compatibility conditions, as well as diverse forms of differential equations. As such, the findings of this research are expected to contribute significantly to the ongoing development of fuzzy analysis and its applications in mathematical modeling and dynamic systems.

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