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Best Simultaneous Approximation in L_p, 0 < p < ∞

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Abstract

It is essential to study the best approximation of a finite multiple target functions in Lebesgue spaces $\boldsymbol{L_p}$ for $0 < p < \infty$. Here we study the approximation of k target functions, $k \geq 2$ and call it Best Simultaneous Approximation (BSA). Also, we introduce some relations between that BSA and the best approximation of one target function. Those definitions are supported by examples that describe the relations as well.

Keywords: Approximation and Simultaneous, L_p

التقريب المتزامن الافضل في الفضاء $\boldsymbol{L_p}$, $0 < p < \infty$

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الخلاصة

من الأهمية بمكان دراسة التقريب الأفضل لعدد منته من الدوال الليبيكية في الفضاء $\boldsymbol{L_p}$, عندما $0 < p < \infty$ ندرس هنا تقريب k من دوال الهدف، $k \geq 2$ ونسميه افضل تقريب متزامن في فضاءات L_p . كما نعرض بعض العلاقات بينه وبين افضل تقريب لدالة هدف واحدة. كما تم دعم تلك التعاريف بأمثلة تصف العلاقات أعلاه.

1. Introduction

The problem of best approximation of function gets great area of study through years, see [1], [2], [3], [4]. In particular, [5] Diaz, J.B. et al, studied best approximation of two continuous functions. In [4] Phillips, G.M. et al generalized the results for L_1 and L_2 normed space. Simultaneous this approximation with the sum norm was taken into account by Ling [6] and [7], [8]. Here , we generalize the BSA of two functions to k arbitrary functions in $L_p, 0 < p < \infty$. Let us introduce some main definitions beginning with the quasi-normed space $L_p[a, b], 0 < p < \infty$, of all measure-functions with the quasi-norm [4]

$$\|f\|_p = \left(\int_a^b |f(x)|^p dx\right)^{1/p}$$

In the following definition, we give a description of the approximation of functions simultaneously,

Definition.1.1. $s^* \in S \subset L_p[a,b]$, is said to be BSA to $f_1, f_2, \dots, f_k \in L_p[a,b]$ iff

$$\inf_{s\in S}\max_i \left\| f_i - s \right\|_p = \max_i \left\| f_i - s^* \right\|_p.$$

2. Best Simultaneous Approximation (BSA)

Now, we present the main theorems that characterize the BSA in L_p.

Theorem 2.1. Let s_1, s_2 is best approximation to $\{f_1, f_2, \dots, f_k\}$. Then $\frac{s_1+s_2}{2}$ is BSA to $\{f_1, f_2, \dots, f_k\}$.

Proof.

Since s_1 and s_2 are best approximation to $\{f_1, f_2, \dots, f_k\}$, so by Definition1.1,

$$\inf_{s\in S}\max_i \left\| f_i - s \right\|_p = \max_i \left\| f_i - s_1 \right\|_p,$$

and

$$\inf_{s\in S}\max_i \left\| f_i - s \right\|_p = \max_i \left\| f_i - s_2 \right\|_p .$$

Thus,

$$\max_i \left\| f_i - \frac{s_1+s_2}{2} \right\|_p \le \frac{1}{2}\max_i \left\| f_i - s_1 \right\|_p + \frac{1}{2}\max_i \left\| f_i - s_2 \right\|_p$$
$$\le \frac{1}{2}\inf_{s\in S}\max_i \left\| f_i - s \right\|_p + \frac{1}{2}\inf_{s\in S}\max_i \left\| f_i - s \right\|_p$$
$$\le \inf_{s\in S}\max_i \left\| f_i - s \right\|_p.$$

So, $\frac{s_1+s_2}{2}$ is BSA of $\{f_1, f_2, \dots, f_k\}$. .

Remark.2.2. It is clear if $\frac{s_1+s_2}{2}$ is best approximation to $\{f_i\}_{i=1}^k$ so, each s_1 and s_2 are BSA to $\{f_i\}_{i=1}^k$.

Now let us prove the following shape preserving simultaneous approximation theorem, call it even simultaneous approximation, which can be defined as, if $\{f_i\}_{i=1}^k$ even function in $L_p[a,b]$, then our simultaneous approximation is also even.

Theorem.2.3. Let $\{f_i\}_{i=1}^k \subset L_p[a,b]$ are even function. Then there s^* even BSA to $\{f_i\}_{i=1}^k$.

Proof.

Since s^* is BSA to the even family of the functions $\{f_i\}_{i=1}^k$. Then,

$$\max_i \left\| f_i - s^* \right\|_p = \max_i \left(\int_a^b |f_i(x) - s^*(x)|^p dx \right)^{1/p}$$

Assume $s^*(x) = s^\wedge(-x)$

$$\inf_{s\in S}\max_i \left\| f_i - s^* \right\|_p = \max_i \left(\int_a^b |f_i(-x) - s^\wedge(-x)|^p dx \right)^{1/p}$$
$$= \max_i \left(\int_a^b |f_i(-x) - s^\wedge(x)|^p dx \right)^{1/p}$$

Since $\{f_i\}_{i=1}^k$ are even function so,

$$\left\| f_i - s^* \right\|_p = \max_i \left(\int_a^b |f_i(x) - s^\wedge(x)|^p dx \right)^{1/p}$$

$$= \inf_{s\in S}\max_i \left\| f_i - s^\wedge \right\|_p.$$

Corollary.2.4. If $s^* \in L_p[a,b]$ is best approximation to each $f_i \in L_p[a,b], i = 1,2,\dots,k$, then s^* is BSA to $\{f_i\}_{i=1}^k \subset L_p[a,b], i = 1,2,\dots,k$.

Proof.

If s^* is best approximation to f_i , $i = 1,2,\dots,k$. $(i.e)$

$$\left\| f_i - s^* \right\|_p = \inf_{s\in S} \left\| f_i - s \right\|_p.$$

This implies

$$\max_i \left\| f_i - s^* \right\|_p = \inf_{s\in S}\max_i \left\| f_i - s \right\|_p.$$

Theorem2.4. If s^* is BSA to $\{f_1, f_2\} \subset L_p[a,b]$, then s^* is best approximation to $\frac{f_1+f_2}{2}$.

Proof.

Since $s^* \in L_p[a,b]$ is BSA to $\{f_1, f_2\}$, then

$$\max_i \left\{ \left\| f_1 - s^* \right\|_p, \left\| f_2 - s^* \right\|_p \right\} = \inf_{s\in S}\max_i \left\{ \left\| f_1 - s \right\|_p, \left\| f_2 - s \right\|_p \right\},$$

So

$$\left\| f_1 - s^* \right\|_p = \inf_{s\in S} \left\| f_1 - s \right\|_p \text{ or } \left\| f_1 - s^* \right\|_p = \inf_{s\in S} \left\| f_2 - s \right\|_p ,$$

and

$$\left\| f_2 - s^* \right\|_p = \inf_{s\in S} \left\| f_2 - s \right\|_p \text{ or } \left\| f_1 - s^* \right\|_p = \inf_{s\in S} \left\| f_1 - s \right\|_p,$$

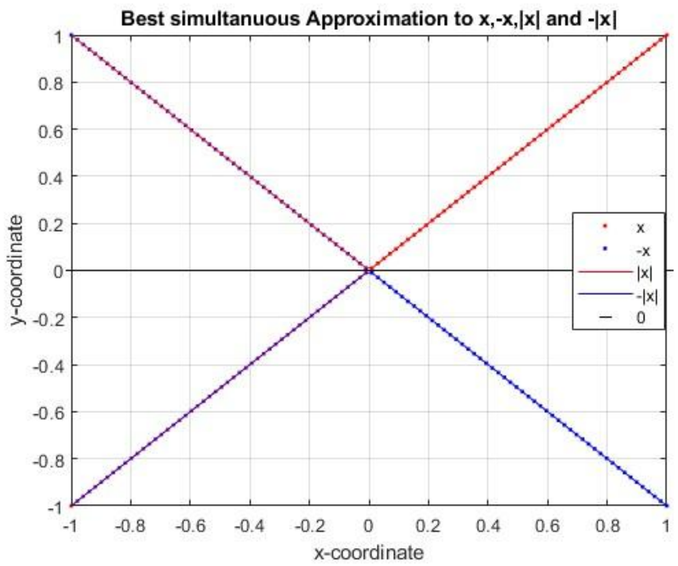
Now

$$\left\| \frac{f_1 + f_2}{2} - s^* \right\|_p = \inf_{s\in S} \left\| \frac{f_1 + f_2}{2} - s \right\|_p$$

$$\begin{aligned}
\left\| \frac{f_1 + f_2}{2} - s^* \right\|_p &= \left\| \frac{f_1 + f_2}{2} - \left(\frac{s^*}{2} + \frac{s^*}{2} \right) \right\|_p \\
&= \left\| \frac{f_1}{2} - \frac{s^*}{2} + \frac{f_2}{2} - \frac{s^*}{2} \right\|_p \\
&\le \tfrac{1}{2} \left\| f_1 - s^* \right\|_p + \tfrac{1}{2} \left\| f_2 - s^* \right\|_p \\
&= \tfrac{1}{2} \inf_{s\in S} \left\| f_1 - s^* \right\|_p + \tfrac{1}{2} \inf_{s\in S} \left\| f_2 - s^* \right\|_p \\
&\le \inf_{s\in S} \left\| \frac{f_1 + f_2}{2} - s \right\|_p.
\end{aligned}$$

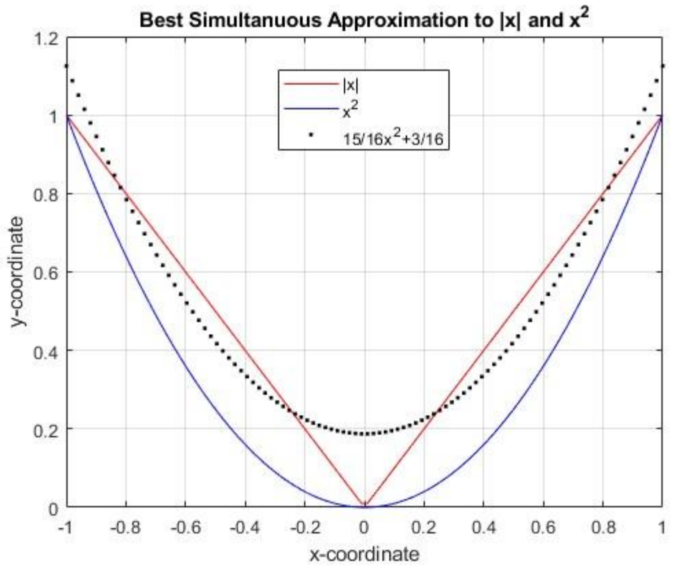
Examples.

1. In $L_\infty[-1,1]$, 0 is the BSA to $\pm x$ and $\pm|x|$ from $\prod_1 = \{a_0 : a_0 \in \mathbb{R}, x \in [-1,1]\}$, as in Figure(1).



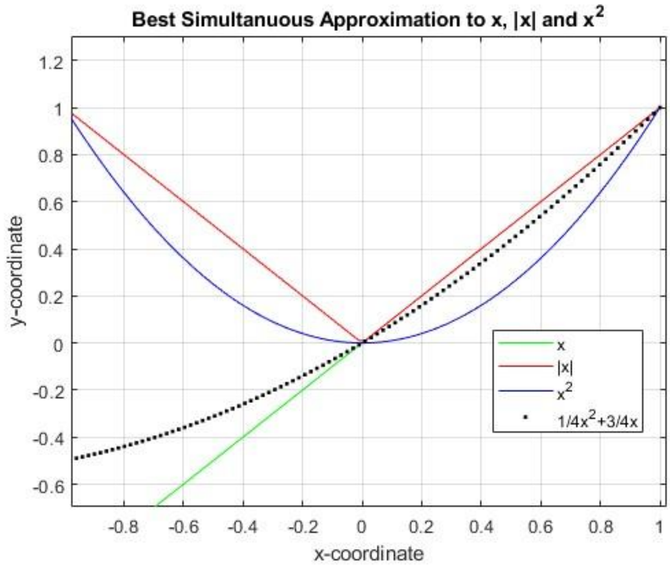
Figure(1) 0 is BSA to $\pm x$ and $\pm|x|$ from $\prod_1$

2. In $L_2[-1,1]$, $\frac{15}{16}x^2+\frac{3}{16}$ is the BSA to $|x|$ and x^2 from $\prod_2=\{a_0+a_1x+a_2x^2: a_i\in \mathbb{R}, x\in[-1,1], i=0,1,2\}$, , as in Figure(2).



Figure(2) $\frac{15}{16}x^2+\frac{3}{16}$ is BSA to $|x|$ and x^2 from $\prod_2$

3. Moreover, in $L_2[-1,1]$, $\frac{1}{4}x^2+\frac{3}{4}x$ is the BSA to $x, |x|$ and x^2 from $\prod_2$, , as in Figure(3).



Figure(3) $\frac{1}{4}x^2 + \frac{3}{4}x$ is BSA to to $x, |x|$ and x^2 from $\prod_2$

3.Conclusions

Lebesgue integrable functions space is a great choice for studying best approximation, for its importance in different fields. In this paper, we study L_p BSA of k functions when $k \geq 2$ in L_p spaces, $0 < p < \infty$. Moreover, we establish some relations among different formulas of best approximation. So that one could get BSA from best approximation and wise versa.

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