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Calculation of Non-Commutative Effects on Heavy Mesons in the Non-Relativistic Limit

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Abstract

The main topic of this article is to consider effect of non-commutative space in the energy spectrum of heavy meson. We have used a nonrelativistic model to obtain the energy shift of low-lying states of a heavy meson with Cornell, and Killingbeck potentials on non-commutative space. Hence, first, we calculated the energy by using the Killingbeck and Cornell potential of the ground state in non-commutative space. With the use of the trial wave function for low-lying states, we found the results are proportional to the two-degree function of the non-commutative (NC) parameter. NC parameter is obtained by comparing the results with experimental hyperfine splitting for bottom onium. It is shown that, at the lowest order of perturbation, the ground state acquires an energy shift that is proportional to the square of the non-commutative length scale We have calculated the hydrogen atom spectrum on curved noncommutative space defined by the commutation relations.

Keywords: Non commutative space; ground state; Killingbeck potential; Cornell potential

حساب تأثيرات الفضاء غير المتبادل على الميزون الثقيل في الحد الغير النسبي

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الخلاصة

يتناول هذا المقال بشكل أساسي تأثير الفضاء غير التبادلي على طيف طاقة الميزون الثقيل وذلك باستخدام نموذجًا غير نسبي لحساب إزاحة الطاقة للحالات الدنيا لميزون الثقيل مع كمونات كورنيل وكيلينغبيك في الفضاء غير التبادلي في البداية حسبنا الطاقة باستخدام كمونات كيلينغبيك وكورنيل للحالة الأرضية في الفضاء غير التبادلي وباستخدام دالة الموجة التجريبية للحالات الدنيا وجدنا أن النتائج تتناسب مع دالة من الدرجة الثانية لمعامل عدم التبادلية (NC). وتم الحصول على معامل (NC) بمقارنة النتائج مع الانفصال الفائق الدقيق التجريبي لذرة الأونيوم السفلية. تبين أنه عند أدنى رتبة من الاضطراب تكتسب الحالة الأرضية إزاحة طاقة تتناسب مع مربع مقياس الطول غير التبادلي وقد قمنا بحساب طيف ذرة الهيدروجين على فضاء غير التبادلي منحني محدد بعلاقات التبادل.

1. Introduction

Field of noncom mutative space and many articles have been presented in this field This study aims to explore non-commutative space. Extensive research has been" conducted in this field, and numerous articles have been published on the topic. This research determines the parameter value of this space [1]. To determine the extent of this parameter, which will be discussed in detail in the next chapter, a series of potentials necessary to describe the work are required. Many potential models are used to study the heavy quarkonium spectrum, and in this section, a limited number of them are examined [2]. The research, in particular, focuses on heavy mesons, often including systems like bottomonium (Upsilon), although this specific focus is not clearly stated in all sections. In this chapter, the selected potentials are mentioned, and their deformation in the non-commutative space is examined [3].Then, using the trial wave function, the energy spectrum of the heavy quarkonium in the non-relativistic state in the non-commutative space is studied. Finally, using two potential models, the parameter value of this space is obtained. Elementary particle physics is a topic that demonstrates the composition of matter and gradually explains its origin. Today, elementary particle physics and astrophysics are closely related, as cosmic space and its events are believed to be the best laboratory for investigating the interactions of elementary particles, especially for reactions not possible on Earth [4].

Fermions are particles that make up materials and possess semi-integer spins. Their classification includes quarks and leptons. Quarks are known to be the major particles that make up hadrons. There are six different types of quarks, each referred to as a flavor up, down, charm, strange, top, and bottom [1]. The six quarks are paired into three generations. The up and down quarks are the lightest and most common in the universe. In contrast, the heavier quarks charm, strange, top, and bottomare unstable and rapidly decay into the lighter up and down quarks [5]. The top quark is, in fact, the heaviest elementary particle. Corresponding to these six quarks, there are six anti-quarks, one antiparticle for each. Three units or colors are found for each quark (green, red, and blue), and each anti-quark contains a negative unit of color charge [6]. The four known fundamental forces in the world are the weak nuclear force, the strong nuclear force, electromagnetism, and gravity. Gravitational force is often negligible in antiparticle interactions due to its very low strength at atomic and subatomic scales [7, 8].

The weak nuclear force is responsible for the beta decay of the nucleus. Its strength is somewhat weaker than that of the strong nuclear force. The range of this force is less than one fathometer; thus, at larger separations, this force can be neglected. However, it is of special importance for better understanding particle behaviour [9, 10]. The coupling constant in weak interactions is of the order of 10^{-7}, and the intermediate particles in these interactions are z^0, w^+, and w^-.The electromagnetic force plays an important role in the interaction between particles and their structure. The electromagnetic force, like gravitational force, has an infinite range, but a phenomenon called electric coating reduces the strength of this force regarding the impact particles in the environment. In processes involving weak interaction, the weak nuclear forces are introduced, and after the destruction of the electron-positron pair, a real boson (z^0) is produced instead of a virtual photon [11]. In classical physics, the electrostatic force between two charged particles decreases with increasing distance. When quarks begin to separate in a pair, the field energy between them increases, creating an extra pair of quarks. Quark confinement states that quarks and anti-quarks are bound together forever, producing hadrons, which are colorless particles [12]. Hadrons, like protons and neutrons, consist of q mesons, and baryons are composed of q .

2. Calculation details

An elementary particle is a particle whose internal structure has not yet been determined. Stem particles known in physics are studied by a theory called the Standard Model of particle physics [4]. These particles are composed of elementary fermions (which make up matter and antimatter) and elementary bosons (which mostly carry the forces of nature). Any particle that consists of several basic particles is a composite particle. CERN, the European Organization for Nuclear Research, is the world's largest particle physics laboratory. Established in 1954 in Geneva, Switzerland, it currently includes twenty European member countries and employs thousands of scientists and engineers. CERN's main activity involves particle accelerators and other infrastructure used for high-energy physics research. CERN's four major detectors are the result of international collaboration. Elementary particles contain elementary fermions that are classified into two main groups: quark and lepton. Quarks are particles that are affected by all four fundamental forces. Initially, particles were categorized by mass. The lightest particles (such as electrons) were called leptons, and the heavier particles (such as protons and neutrons) were called baryons. Intermediate particles were called mesons (like pions).

Mesons are a group of composite particles with an internal structure. Mesons participate in strong interactions and have integer spin; therefore, statistically, these particles belong to the group of bosons and follow Bose-Einstein statistics. Mesons are mainly produced by strong interactions and decay to other mesons or leptons through strong, weak, and electromagnetic interactions. Baryons are the third group of particles, consisting of atomic components (protons and neutrons) and being heavier than other groups. Leptons are particles that are not affected by strong nuclear forces. Basic bosons are divided into two categories modular bosons and fence bosons.

The main objective of this research is to investigate the wave function at the origin (Psi(0)) for the S-wave bound state of heavy mesons (quarkonium, e.g., c and b systems). This quantity is crucial because it is essential for calculating hyperfine splitting and decay amplitudes of these heavy mesons Heavy quarkonium properties are known to be sensitive to medium effects such as temperature, magnetic fields, and anisotropic plasmas. Recent studies have investigated quarkonium evolution, dissociation, and binding behavior under extreme conditions using lattice QCD and effective potential approaches[13-16].

The study also critically compares different theoretical methods namely, the non-relativistic potential model, numerical solutions (like the Schrödinger equation), the perturbative approach, and the variational method highlighting the deficiencies of each (such as divergences in the perturbative method or the inability of numerical methods to yield analytical interpretations), especially when dealing with singular potentials (e.g., $1/r^3$terms). Ultimately, the research advocates for using the variational method or exploring other non-relativistic potentials, like the Killingbeck potentials, to accurately obtain the wave function and better capture the "real physics" of heavy meson spectroscopy.

3. Results

3.1 Effects of Displaced Space on Binocular Potentials

Noncommutative quantum mechanics has been extensively studied as a theoretical extension of conventional quantum mechanics, motivated by developments in quantum gravity and string theory. Comprehensive reviews and monographs have discussed the mathematical structure, symmetry properties, and physical implications of noncommutative spaces, as well as their applications to quantum and high-energy physics [13, 17-19]. In recent years, concern to perusal quantum theories in modified spaces named noncom mutative space as mentioned in the previous session and determined by noncom mutative coordinates has been renewed [20]. This resurgence is motivated by string theory, where it has been demonstrated that non-commutative field theories naturally appear in a low-energy limit of string models [21]. Furthermore, the objective impacts of a modified space can only be tested at the string length scale or around the grand unification gauge. Consequently, non-commutative models explore the frontiers of ultra-high energy physics where established field theories are no longer sufficient [22]. The primary approach involves extrapolating a few well-established models to these extremes. The current bounds for the non-commutative scale range from the most exact considerations at particle colliders down to the classical limit [23].

Several studies have been conducted in this scope, such as the toy model of a harmonic oscillator on non-commutative space [24], coherent states [25], phenomenological aspects of non-commutative space in the Hydrogen atom spectrum, the Lamb shift, and the spectrum of heavy quarkonium with Cornell potential. In the following, the model is expanded to include Cornell and Killingbeck potentials dependent on the non-commutative space [23]. Recent studies have investigated the effects of spatial noncommutativity on quarkonium systems, demonstrating that noncommutative geometry can induce measurable corrections to mass spectra and wave functions. In particular, variational and perturbative approaches have been successfully applied to analyze quarkonium masses in three-dimensional noncommutative space, revealing a quadratic dependence of energy corrections on the noncommutative parameter [26, 27]. Noncommutative effects on phenomenological potentials have been explored for heavy and heavy–light meson systems. Modified potentials, such as the Yukawa-type interactions formulated within noncommutative quantum mechanics, have been shown to significantly affect bound-state properties and energy spectra [28, 29].

3.2 KillingBeck Potential

We consider the potential required to calculate the displacement space parameter as the following equation:

$$V(r) = Ar^2 + Br - \frac{c}{r} \qquad (1)$$

With, A= 0.143 GeV^2, B= 0.465 GeV^2, C= 0.284 GeV^2, $m_b = 4.68\ \mathrm{GeV}$. Using the wave function

$$\psi = Ne^{-ar^b} \qquad (2)$$

As follows, we select the not-difficult trial wave function in which there is just one vibrational parameter to study the 1S state of $b\bar{c}$. The general form of such trial wave function is written as in which the values of b and the normalization coefficient are known and the constant value of a can also be obtained as follows:

$$b = 1,2,1/5,1/3 \quad , \quad N^2 = \frac{b(2a)^{\frac{3}{b}}}{4\pi\Gamma(\frac{3}{b})} \tag{3}$$

N is the normalization constant, a indicates the vibrational parameter that can be steady by minimizing the anticipation amount of Schrödinger Hamiltonian and b is the parameter of the model that distinguishes the kind of the chosen trial wave function. As follows, we chose the four trial wave functions below.

b = 1, namely $\psi = Ne^{-ar^1}$ (4)

(The wave function is named as Hydrogen wave function and is an exponential wave function). Also, this function is the resolution of the model of Coulomb potential.

b = 2, namely $\psi = Ne^{-ar\,2}$ (5)

(This function is the wave function of the Harmonic oscillator and it is a Gaussian wave function).

$b = \frac{3}{2}$, namely $\psi = Ne^{-ar1.5}$ (6)

The mention function was used by Gupta [6].

$b = \frac{4}{3}$, namely $\psi = Ne^{-ar\frac{4}{3}}$. (7)

This function is a new trial wave function that was recently proposed.

The mentioned trial wave function can be with two or more vibrational parameters and then will have different forms. The straightway one is to multiply by a polynomial of r, as:

$$\psi = \left(c_0 + c_1 r + c_2 r^2 + ... + c_n r^n\right) e^{-ar^b} \tag{8}$$

With a and c_1, c_2 can are vibrational parameters, and c_0 may be constant by the normalization condition .

In order to understand the precision of the vibrational results in the Killing beck Potential case, we account for one quantity by applying four different mentioned trial wave functions and the corresponding results by also dissolving numerically the Schrödinger equation. Therefore, the average value of Hamilton concerning the sum of kinetic energy and potential can be written as follows:

$$\langle H \rangle = \langle T \rangle + \langle V \rangle \tag{9}$$

In the mentioned formula, the average kinetic energy can be calculated as follows:

$$\langle T \rangle = \left\langle \frac{P^2}{2\mu} \right\rangle = \left\langle \psi \left| \frac{-\hbar^2 \nabla^2}{2\mu} \right| \psi \right\rangle \tag{10}$$

Using the definition of ∇^2:

$$\nabla^2 = \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} \tag{11}$$

Which helps in solving the following integral from the following mathematical relation.

$$\int_0^\infty r^n \, e^{-ar} \; dr = \frac{\Gamma(\frac{n+1}{b})}{b\, a^{\frac{n+1}{b}}} \tag{12}$$

Finally, the following relationships can be found for potential energy and kinetic energy:

$$\langle T \rangle = \frac{(2a)^{\frac{2}{b}} \; b^2 \; \Gamma(2 + \frac{1}{b})}{8\mu \, \Gamma(\frac{3}{b})} \tag{13}$$

Then we have:

$$\langle V \rangle = \langle \psi \,|\, V \,|\, \psi \rangle = \left\langle \psi \left| Ar^2 + Br - \frac{c}{r} \right| \psi \right\rangle = \left\langle \psi \left| Ar^2 \right| \psi \right\rangle + \langle \psi \,|\, Br \,|\psi \rangle - \left\langle \psi \,|\, \frac{c}{r} \,|\psi \right\rangle \tag{14}$$

By inserting the value of the wave function in the relation we have:

$$\langle V \rangle = N^2 \int e^{-ar^b} (Ar^2 + Br - \frac{c}{r})\, e^{-ar^b}\, 4\pi r^2 dr = \frac{b(2a)^{\frac{3}{b}}\; 4\pi}{4\pi\,\Gamma(\frac{3}{b})} \int_0^\infty [Ar^4 + Br^3 - Cr)] e^{-2ar^b} \; dr \tag{15}$$

And finally, we will have:

$$\langle V \rangle = \frac{b\;(2a)^{\frac{3}{b}}}{\Gamma(\frac{3}{b})} [A \frac{\Gamma(\frac{5}{b})}{\Gamma\;(2a)^{\frac{5}{b}}} + B\, \frac{\Gamma(\frac{4}{b})}{b\;(2a)^{\frac{4}{b}}} - C\, \frac{\Gamma(\frac{2}{b})}{b\;(2a)^{\frac{2}{b}}} \tag{16}$$

Using the relationships obtained for kinetic energy and potential we have:

$$\langle H\rangle=\langle T\rangle+\langle V\rangle=\frac{(2a)^{\frac{2}{b}}\,b^2\,\Gamma(2+\frac{1}{b})}{8\mu\,\Gamma(\frac{3}{b})}+\frac{b(2a)^{\frac{3}{b}}}{\Gamma(\frac{3}{b})}[A\frac{\Gamma(\frac{5}{b})}{b\,(2a)^{\frac{5}{b}}}+B\,\frac{\Gamma(\frac{4}{b})}{b\,(2a)^{\frac{4}{b}}}-C\frac{\Gamma(\frac{2}{b})}{b\,(2a)^{\frac{2}{b}}} \tag{17}$$

To obtain the values of a, we derive the relation obtained for Hamiltonian from this variable and set it to zero:

$$\frac{\partial\,\langle H\rangle}{\partial a}=0\;\Rightarrow$$

$$\frac{\partial}{\partial a}\,\frac{(2a)^{\frac{2}{b}}\,b^2\,\Gamma(2+\frac{1}{b})}{8\mu\,\Gamma(\frac{3}{b})}=\frac{b^2\,\Gamma(2+\frac{1}{b})}{8\mu\,\Gamma(\frac{3}{b})}\left[\frac{2}{b}\times 2\times(2a)^{\frac{2}{b}-1}\right]$$

$$\Rightarrow\frac{b\,(2a)^{\frac{3}{b}}\times b\,(2a)^{\frac{-5}{b}}}{\Gamma(\frac{3}{b})}\,A\Gamma(\frac{5}{b})+\frac{b\,(2a)^{\frac{3}{b}}\times b\,(2a)^{\frac{-4}{b}}}{\Gamma(\frac{3}{b})}\,B\Gamma(\frac{4}{b})+$$

$$\frac{b\,(2a)^{\frac{3}{b}}\times b\,(2a)^{\frac{-2}{b}}}{\Gamma(\frac{3}{b})}-C\,\Gamma(\frac{2}{b})$$

$$=\frac{b}{\Gamma\,(\frac{3b}{b})}\left[(2a)^{\frac{-2}{b}}\,A\Gamma(\frac{5}{b})+(2a)^{\frac{-1}{b}}\,B\Gamma(\frac{4}{b})-(2a)^{\frac{1}{b}}\,C\Gamma(\frac{2}{b})\right]$$

$$=\frac{b}{\Gamma(\frac{3}{b})}\Bigg[A\Gamma(\frac{5}{b})\,(\frac{-2}{b})\times 2\times(2a)^{\frac{-2}{b}-1}+B\Gamma(\frac{4}{b})\,(\frac{-1}{b})\times 2\times(2a)^{\frac{-1}{b}-1}$$

$$-C\,\Gamma(\frac{2}{b})\,(\frac{1}{b})\times(2a)^{\frac{1}{b}-1}\times 2\Bigg]=0 \tag{18}$$

For the part related to T we have:

$$\frac{b^2\,\Gamma(2+\frac{1}{b})}{8\mu\,\Gamma(\frac{3}{b})}\left[\frac{2}{b}\times 2\times(2a)^{\frac{2}{b}-1}\times(2a)^{\frac{2}{b}+1}\right.=\frac{b^2\,\Gamma(2+\frac{1}{b})}{4\mu\,\Gamma(\frac{3}{b})}\,(2a)^{\frac{4}{b}} \tag{19}$$

Then, for the part related to V we have:

$$\frac{b}{\Gamma(\frac{3}{b})}\left[(2a)^{-\frac{2}{b}}\times(2a)^{\frac{2}{b}+1}A\Gamma(\frac{5}{b})+(2a)^{\frac{-1}{b}}+\times(2a)^{\frac{2}{b}+1}B\Gamma(\frac{4}{b})-(2a)^{\frac{1}{b}}\right.$$
$$\times(2a)^{\frac{2}{b}+1}C\Gamma(\frac{2}{b})=0 \tag{20}$$

And finally we will have:

$$\frac{\partial\langle H\rangle}{\partial a}=0\Rightarrow\frac{b^2}{4\mu}(2a)^{\frac{4}{b}}\frac{\Gamma(2+\frac{1}{b})}{\Gamma(\frac{3}{b})}-2A\frac{\Gamma(\frac{5}{b})}{\Gamma(\frac{3}{b})}-B\frac{\Gamma(\frac{4}{b})}{\Gamma(\frac{3}{b})}\times$$
$$(2a)^{\frac{1}{b}}-C(2a)^{\frac{3}{b}}\frac{\Gamma(\frac{2}{b})}{\Gamma(\frac{3}{b})}=0 \tag{21}$$

Finally, we come to the following relation:

$$\Rightarrow\frac{b^2}{4\mu}x^4\Gamma(2+b)-2A\Gamma(\frac{5}{6})-B\,\Gamma(\frac{4}{b})x-C(x^3)\,\Gamma(\frac{2}{b})=0 \tag{22}$$

The values obtained for the parameter are given in the table below.

Table 1- Different values of Vibrational parameter.

Parameter (b)	**Corresponding value of (a)**
$b=1$	$a=1.346$
$b=2$	$a=0.542$
$b=1.5$	$a=0.829$
$b=1.3$	$a=0.995$

Using the values of a and b, the values of N and finally are obtained. We now consider the altered shape of the potential in the displacement space. For this purpose, it must first be noted that the potential is a function of the place vector. For this purpose, we must first obtain the modified form of the place vector in the displacement space. The result is as follow

$$x = \hat{x} + \frac{\theta ij}{2\hbar} P \longrightarrow \hat{x} = x - \frac{\theta ij}{2\hbar} P \qquad (23)$$

By placing the above relation in the kernel potential we will have:

$$V(r) = Ar^2 + Br - \frac{C}{r} \quad , \quad V^{(NC)} = A\hat{r}^2 + B\hat{r} - \frac{C}{\hat{r}} \qquad (24)$$

Now we need to get the values with respect to the above conversion for the NC space:

$$\hat{r}^2 = \hat{r}.\hat{r} = \hat{r} = \sqrt{\hat{r}.\hat{r}} = \sqrt{\hat{x}.\hat{x}} = \sqrt{(x_i - \frac{P_i}{2\hbar}\theta_{ij})(x_j - \frac{P_j}{2\hbar}\theta_{ik})} \qquad (25)$$

$$\hat{r} = \sqrt{x_i x_j - \frac{1}{2\hbar}\theta_{ij}.L + \frac{1}{4\hbar^2} P_i P_j \theta_{ij}\theta_{ik} + \dots}$$

$$\hat{r} = r\left[1 - \frac{1}{2\hbar r^2}\theta.L + \frac{1}{4\hbar^2}\frac{1}{r^2} P_i P_j \theta_{ij}\theta_{ik} + \dots\right]^{\frac{1}{2}}$$

$$\hat{r} = r\left[1 + \frac{1}{2r^2}\left[-\frac{1}{2\hbar}\theta.L + \dots\right] + \dots\right]$$

Now that the desired relationships have been obtained, the modified form of the potential can be calculated:

$$V^{(NC)} = A\hat{r}^2 + B\hat{r} - \frac{C}{\hat{r}}$$

$$\Rightarrow A\left[(r^2(1 - \frac{1}{r^2}(\frac{1}{2\hbar}L.\theta - \frac{1}{4\hbar^2}\theta_{ij}\theta_{ik}P_jP_k + Br(1 - \frac{1}{r^2}(\frac{1}{4\hbar}L.\theta - \frac{1}{8\hbar^2}\theta_{ij}\theta_{ik}P_jP_k)\right]$$

$$-C\left[\frac{1}{r}(1 + \frac{1}{r^2}(\frac{1}{4\hbar}L.\theta - \frac{1}{8\hbar^2}\theta_{ij}\theta_{ik}P_jP_k + \dots)\right] \qquad (26)$$

$$V = Ar^2 - \frac{A}{2\hbar}L.\theta + \frac{A}{4\hbar^2}\theta_{ij}\theta_{ik}P_jP_k + Br - \frac{B}{4r\hbar}L.\theta + \frac{B}{8r\hbar^2}P_{ij}P_{ik}\theta_{ik}\theta_{ij} - \frac{C}{r} +$$

$$\frac{C}{4\hbar r^3}L.\theta - \frac{C}{8r^3\hbar^2}\theta_{ij}\theta_{ik}P_jP_k =$$

$$V = Ar^2 - \frac{A}{2\hbar}L.\theta + \frac{A}{4\hbar^2}\theta_{ij}\theta_{ik}P_jP_k + Br - \frac{B}{4r\hbar}L.\theta + \frac{B}{8r\hbar^2}P_{ij}P_{ik}\theta_{ik}\theta_{ij} - \frac{C}{r} +$$

$$\frac{C}{4\hbar r^3}L.\theta - \frac{C}{8r^3\hbar^2}\theta_{ij}\theta_{ik}P_jP_k =$$

$$V = Ar^2 - \frac{A}{2\hbar} L.\theta + \frac{A}{4\hbar^2}\ \theta_{ij}\theta_{ik}P_jP_k + Br - \frac{B}{4r\hbar} L.\theta + \frac{B}{8r\hbar^2} P_{ij}P_{ik}\theta_{ik}\theta_{ij} - \frac{C}{r} +$$

$$\frac{C}{4\hbar r^3} L.\theta - \frac{C}{8r^3\hbar^2}\theta_{ij}\theta_{ik}\ P_jP_k$$

The resulting V consists of two parts, one called the pre-defined V and the new part called the perturbation potential.

$$V^{NC} = L.\theta\ (\frac{A}{2\hbar} - \frac{B}{4r\hbar} + \frac{C}{4r^3\hbar}) + \theta_{ij}\theta_{ik}P_jP_k\left[\frac{A}{4\hbar^2} + \frac{B}{8r\hbar^2} - \frac{C}{8r^3\hbar^2}\right] \tag{27}$$

$$V^{NC} = \frac{1}{4}\ (P_x^2 + P_y^2)\ \theta^2\left(\frac{A}{4\hbar^2} + \frac{B}{8r\hbar^2} - \frac{C}{8r^3\hbar^2}\right)$$

According to the obtained disturbance V, the disturbance energy can be calculated for the system.

$$E_{NC} = \langle\psi|V_{NC}|\psi\rangle = \left\langle\psi\ \left|\ \frac{1}{4}\ (P_x^2 + P_y^2)\ \theta^2\left(\frac{A}{4\hbar^2} + \frac{B}{8r\hbar^2} - \frac{C}{8r^3\hbar^2}\right)\right|\psi\right\rangle$$

$$= \frac{\theta^2}{4}\ \frac{A}{4\hbar^2}\ \left\langle\psi\left|P_x^2 + P_y^2\right|\psi\right\rangle + \frac{Br^2}{4\times 8\hbar^2}|\psi|\ \frac{\left(P_x^2 + P_y^2\right)}{r}\ |\ \psi\rangle - \frac{C}{4\times 8\hbar^2}\ \left\langle\psi\ \left|\frac{P_x^2 + P_y^2}{r^2}\right|\psi\right\rangle$$

Using the opposite approximation:

$$\frac{\langle P_x\rangle^2}{r^n} = \frac{\langle P_y\rangle^2}{r^n} = \frac{\langle P_z\rangle^2}{r^n} = \frac{1}{3} < \frac{P^2}{r^n} > \tag{28}$$

We have:

$$E_{NC} = \frac{\theta^2}{32\hbar^2}\left[A\times\frac{2}{3}\left\langle P^2\right\rangle + B\times\frac{2}{3}\left\langle\frac{P^2}{r}\right\rangle + C\times\frac{2}{3}\left\langle\frac{P^2}{r^3}\right\rangle\right] \tag{29}$$

To calculate the averages, we do the following:

$$\left\langle P^2\right\rangle = \frac{b\ (2a)^{\frac{3}{b}}}{\Gamma\ (\frac{3}{b})}\int\left[(ab + ab^2)r^b - a^2b^2\, r^{2b}\right]e^{-2a\ r^b}\ \ dr$$

$$\left\langle\frac{P^2}{r}\right\rangle = \frac{b\ (2a)^{\frac{3}{b}}}{\Gamma\ (\frac{3}{b})}\int e^{-2a\,r^b}\left((ab + ab^2)\,r^{b-1} - a^2b^2\, r^{2b-1}\right)dr \tag{30}$$

$$\left\langle\frac{P^2}{r^3}\right\rangle = \frac{b\ (2a)^{\frac{3}{b}}}{\Gamma\ (\frac{3}{b})}\int e^{-2a\,r^b}\left[(ab + ab^2)\,r^{b-3} - a^2b^2\, r^{2b-3}\right]dr$$

For different values of parameter b, we calculate the desired value and from the obtained equations, the energy value of the displacement space can be calculated for the four values of parameter b. The results are the average value based on an analysis of both Roth and pare harmonium spectra. The mass splitting between Roth and Para $b\bar{c}$ or the splitting of energy levels, according to the experimental results can be generally written as:

$$M\left(s_1^3\right)-M\left(s_1^0\right)=-\frac{32\pi\alpha_s}{9m_1m_2}\left|\psi\left(0\right)\right|^2 \qquad (31)$$

and causes a 0.049 MeV [30, 31] mass difference between the two states. Where $m_1$and m_2 are the masses of a bottom quark. Finally, by comparing the amount of energy obtained with hyperfine splitting for the bottomonium mason, the value of the parameter can be obtained, which results are presented in the ground state of this meson and using the Killing beck potential in Table 2-3. We used a trial wave function to evaluate the ground state energy of b $\bar{c}$ bound state on noncom mutative space with Killing beck and Cornell potentials for the ground state, in nonrelativistic limit.
Finally, the values of the displacement space parameter are as follows:

Table 2- Different values of energy and noncom mutative parameter

Value of (b)	**Upper limit of Energy (Θ)**	**Value of E_{NC}**
$b=1$	$\theta \le (\frac{1}{1.204} Gev)^{-2}$	$E_{NC}=0.10297\theta^2$
$b=2$	$\theta \le (\frac{1}{0.693} Gev)^{-2}$	$E_{NC}=0.01131\theta^2$
$b=1/3$	$\theta \le (\frac{1}{0.989} Gev)^{-2}$	$E_{NC}=0.04698\theta^2$
$b=1/5$	$\theta \le (\frac{1}{0.877} Gev)^{-2}$	$E_{NC}=0.02941\theta^2$

3.3 Cornell Potential

Now for the first ground state and for a given potential we want to get the value of the noncom mutative parameter.

$$V=\frac{-4}{3}\frac{\alpha_s}{r}+Kr \qquad (32)$$

With, $\alpha_s=0.39$, $m_b=5.18$ GeV, $k=\frac{1}{2.34^2}$ and r is the interquark interval. Also, α_s depends on r but this is weak and we neglect this relation. Moreover, k is the confinement constant. One gluon exchange between the quark-antiquark systems which overcomes short spaces is caused by the first part of Cornell's potential. The second term of Eq. (29.) is for quark confinement at great distances. In addition, each potential that is used for the study of heavy mesons should have two defined forms of strong interaction, which are asymptotic freedom and confinement .As a previous calculation, we can obtain the NC parameter for Cornell potential. Where k is a constant value of about 16 t, the color coefficient is 4/3, and α_s= 0.39 is the quark-gluon coupling constant .To calculate the value of the noncom mutative space operator parameter using the cornel potential, given that the value of this parameter is calculated for its ground state of harmonium mason using the trial wave function of relation, and performing calculations similar to those performed for the Killingbeck potential means that we will finally compute the value of the parameter, the results of which are given in Tables 3and 4. Experimental measurements of quarkonium states at the LHC provide precise data on mass spectra and hyperfine splittings, offering stringent tests for theoretical models and allowing constraints to be placed on physics beyond the standard framework[16].

Table 3- Different values of Vibrational parameter.

Vibrational parameter (b)	Value of (a)
$b = 1$	$a = 1.028$
$b = 2$	$a = 0.320$
$b = 1.5$	$a = 0.566$
$b = 1.3$	$a = 0.714$

Table 4- Different values of noncom mutative parameter.

Value of (b)	Upper limit of Energy (Θ)	Value of E_{NC}
$b = 1$	$\theta \le (\frac{1}{1.090} Gev)^{-2}$	$E_{NC} = 0.0692\theta^2$
$b = 2$	$\theta \le (\frac{1}{0.566} Gev)^{-2}$	$E_{NC} = 0.00504\theta^2$
$b = 1/3$	$\theta \le (\frac{1}{0.879} Gev)^{-2}$	$E_{NC} = 0.02929\theta^2$
$b = 1/5$	$\theta \le (\frac{1}{0.676} Gev)^{-2}$	$E_{NC} = 0.01696\theta^2$

Given the statement obtained for perturbation energy, we saw that its value is a quadratic function of the noncom mutative parameter, if we want to attribute a numerical value to this parameter as well, we must compare the amount of perturbation energy with the amount of hyperfine splitting, and we also know that hyperfine splitting has different values for the base state. In both cases, we have to consider the amount of perturbation energy equal to or less than the amount of hyperfine splitting to get the parameter value. In the base state, the value of this hyperfine splitting for the bottomonium meson is 0.049 GeV. By calculations that are done, the parameter value is obtained for all of the a and b values which are shown in T|able 2 for both states. Comparing the results with other work done in this field, we can say that the values obtained are in agreement with the values obtained in other studies. Our results based on a simple nonrelativistic model are compatible with other results based on other more sophisticated models and analyses reported in the literature

4. Conclusion

In recent years, the wave function (Psi(0)) of a heavy meson, quarkonium, or quark-antiquark system at the origin for the S-wave bound state has once again renewed the attention of researchers. The reason for this focus is that it is a very significant quantum quantity for considering hyperfine splitting and is essential for assessing the generation and decay amplitude of heavy mesons. For instance, the non-relativistic potential model in Refs. showed the numerical conclusion of the wave function at the origin of the S-wave for heavy mesons as c , b , and a systems and in contrast to those gained in several "successful" potential models.

In addition, there are enumerated potentials that can be used to solve bound state problems analytically, such as the Coulomb and the harmonic oscillator potentials. Also, if the bound state problems are not analytically soluble, one can use approximations for solving these problems. For solving the Schrödinger equation numerically, one can apply a powerful method that may offer great accuracy. Although the numerical method is not perfect and has several deficiencies, one of these defects is that the numerical method cannot render an analytical interpretation for further discussion.

Furthermore, the numerical method for the focal potential is existent for V(r) that has a singularity less than $1/r^2$where r approximates 0. Thus, it fails when $1/r^3$ exists in the potential. Also, conflict often arises in computing the fine-splitting of the P state. In addition to the variational method, the perturbative approach is another extensively used approximation method. Due to divergence, the practical applicability of the perturbative expansion is limited in many cases. Also, the perturbation parameter intervals must comply with the isotropy prescriptions. Furthermore, the behavior of wave functions in this method is harder to ascertain than that for energy eigenvalues. It must be mentioned that it is not easy to obtain the wave function correct to the order of v^2/c^2, as was published by authors in Ref. Because the Hamiltonian in the non-relativistic limit requires the perturbative correction to the wave function to be rendered by an infinite sum over all states. For instance, in the case of positronium, this sum is infinite or diverges because the potential at the origin is divergent.

In addition, for the singular $1/r^3$ in potential models, this method is successful and mostly used. The splitting of the energy level can be calculated by solving an integral, which may diverge, leading to an exotic result that does not tally with real physics . Therefore, the results show that the perturbative computation is not always reliable. Also, for a physical state that may be extended experimentally, a derived resolvent must be matched theoretically, assuming the model is true. To obtain this, for instance, the energy levels of the triplet P states of the charmonium meson should have certain gauge amounts. Because of the non-scalable non-commutative QCD effect, one must rely on particular models theoretically. The quark potential model in the non-relativistic limit is one of these models. With the 3-diagram assessment and the estimation in the non-relativistic limit, a potential term of $1/r^3$ becomes visible. Also, in the framework of the perturbative method, collecting all the high-order diagrams in the perturbative expansion and all the high-order phrases in the non-relativistic reduction is complex. In this way, the question is how to capture the great quantity of real physics.

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