Sumer Journal for Pure Science (ISSN: 2790-7031)
2nd International Scientific Conference on Pure and Medical Sciences/University of Sumer 2024

Truncated Exponentiated Lomax Distribution with Some Important Properties

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Abstract

In this paper, we introduce a new three-parameter of continuous distributions based on interval [0,1], the Truncated exponentiated Lomax (TEL) distribution, is presented and studied. Some Probabilistic properties are examined, The cumulative distribution function, the rth moment, the median, the characteristic function, the hazard rate function and the reliability function are obtained for the distribution under consideration. So maximum likelihood estimation is discussed. It is common knowledge that an object breaks down when the stress it experiences surpasses its matching strength. Strength can be defined as "resistance to failure" in this sense. A strong design always has a strength greater than the anticipated stress. Stress/strength is a definition of the safety factor in terms of stress and strength.. So, Here, the stress-strength model for the Truncated Exponentiated Lomax (TEL) distribution will be generated using various parameters. The Shannon entropy will be obtained.

Keywords: Lomax distribution, exponentiated Lomax distribution, Truncated exponentiated Lomax distribution, Statistical properties, Stress strength, Shannon Entropies.

توزيع لوماكس الأسي المبتور مع بعض الخصائص المهمة

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الخلاصة

في هذا البحث، قمنا بتقديم توزيع مستمر بثلاثة معلمات معرف على الفترة [0,1]، وتم عرض ودراسة توزيع لوماكس الأسي المقطوع .و ايجاد بعض الخصائص الاحتمالية المهمة، مثل دالة التوزيع التراكمي، والعزم ، والوسيط، والدالة المميزة، ودالة المعولية، ودالة معدل الخطر للتوزيع . ودالة الامكان الاعظم. ومن المعروف أن أي عنصر يفشل عندما يتجاوز الضغط الذي يتعرض له القوة المقابلة .وبهذا المعنى، يمكن النظر إلى القوة على أنها "مقاومة الفشل ."ممارسات التصميم الجيدة هي أن القوة تكون دائمًا أكبر من الضغط المتوقع .يمكن تعريف عامل الأمان من حيث الإجهاد والقوة على أنه الإجهاد / القوة .لذلك، سيتم هنا استخلاص نموذج قوة ضغط توزيع Lomax (TEL)المقتطع مع معلمات مختلفة, كذلك تم اشتقاق دالة إنتروبيا شانون.

الكلمات المفتاحية: توزيع لوماكس، توزيع لوماكس الأسي، توزيع لوماكس الأسي المقتطع، الخصائص الإحصائية، قوة الإجهاد، إنتروبيات شانون.

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1. Introduction

A random variable T follows a Lomax distribution, denoted $T\_L(t)$, if its cumulative distribution function (cdf) and probability density function (pdf) are given by [1]

$$F(t) = 1 - \left(1 + \frac{t}{\beta}\right)^{-\alpha} \qquad t > 0, \alpha, \beta > 0 \qquad (1)$$

$$f(t) = \frac{\alpha}{\beta}\left(1 + \frac{t}{\beta}\right)^{-(\alpha+1)} \qquad t > 0, \quad \alpha, \beta > 0 \qquad (2)$$

Studies about Lomax distribution have been discussed by several authors. The exponentiated Lomax geometric (*ELG*) is introduced by Hassan and Allah [2]. exponentiated Lomax Poisson, Lomax-logarithm, and extended Lomax Poisson distributions have been given, respectively by Ramos et al. [3]. A new family of distributions called double Lomax distribution introduced by Bindu and Sangita [4]. beta Lomax, Kumaraswamy Lomax and McDonald Lomax distributions suggested by Lemonte and Cordeiro [5]. Ghitany et al. [6], introduced Marshall-Olkin extended Lomax. Extended Poisson Lomax Distribution introduced by Hamed [7].

A random variable T is said to be have an exponentiated distribution, the comulative distribution function (cdf) and probability density function (pdf) are provided by:

$G(t;\theta) = H(t)^{\theta}$ and $g(t;\theta) = \theta h(t) H(t)^{\theta-1}$

Suppose that G(t) and g(t) represent the cdf and pdf of the Lomax distribution.

2. The transmuted exponentiated Lomax distribution

A random variable T follows a transmuted exponentiated Lomax (TEL) distribution, the probability density function (pdf) and cumulative distribution function (cdf) are provided by:

$$G(\alpha,\beta;\, t) = \left[1 - \left(\frac{t}{\beta} + 1\right)^{-\alpha}\right]^{\theta} \qquad t > 0, \alpha, \beta > 0 \qquad (3)$$

$$g(\alpha,\beta;t) = \frac{\theta\alpha}{\beta}\left(\frac{t}{\beta} + 1\right)^{-(\alpha+1)}\left[1 - \left(\frac{t}{\beta} + 1\right)^{-\alpha}\right]^{\theta-1}, t > 0, \alpha, \beta > 0 \qquad (4)$$

2.1 Truncated Distributions

In this section, a new presented truncated transmuted Exponentiated Lomax (TEL) distribution, based on the interval [0, 1].

Also suppose that G(t) and g(t) are the cdf and pdf of Truncated Exponentiated Lomax distribution on the interval $[0,1]$. So, this is done through the following.

$$F(t)_T = \frac{G(t) - G(0)}{G(1) - G(0)}\; ; 0 < t < 1 \qquad (5)$$

$$f(t)_T = \frac{g(t)}{G(1)} \qquad (6)$$

Suppose that G(t) and g(t) in (5) and (6) represent the cdf and pdf of the truncated exponentiated Lomax distribution that are given in (3) and (4) with three parameters . Then, a new distribution named Truncated Exponentiated Lomax (TEL) distribution is introduced. The cdf and pdf of are given respectively as

$$F(t)_{TEL} = \frac{\left[1-\left(\frac{t}{\beta}+1\right)^{-\alpha}\right]^{\theta}}{\left[1-\left(\frac{\beta}{1+\beta}\right)^{\alpha}\right]^{\theta}}, 0<t<1, \alpha,\beta>0 \quad (7)$$

$$f(t)_{TTEL} = \frac{\frac{\alpha\theta}{\beta}\left(1+\frac{t}{\beta}\right)^{-(\alpha+1)}\left[1-\left(1+\frac{t}{\beta}\right)^{-\alpha}\right]^{\theta-1}}{\left[1-\left(\frac{\beta}{1+\beta}\right)^{\alpha}\right]^{\theta}}, 0<t<1, \alpha,\beta>0 \quad (8)$$

2.2 Reliability Analysis for the TEL Distribution

Utilizing equations (7) and (8), the reliability metrics for the TEL distribution, including the (reliability, hazard, cumulative hazard and reverse hazard) functions, can be readily computed as follows:

$$\xi_1(t)_{TEL} = 1 - F(t)_{TEL} = 1 - \left[\frac{1-\left(\frac{t}{\beta}+1\right)^{-\alpha}}{1-\left(\frac{\beta}{1+\beta}\right)^{\alpha}}\right]^{\theta} \quad (9)$$

$$\xi_2(t)_{TEL} = \frac{f(t)_{TEL}}{1-F(t)_{TEL}} = \frac{\frac{\alpha\theta}{\beta}\left(\frac{t}{\beta}+1\right)^{-(\alpha+1)}\left[1-\left(\frac{t}{\beta}+1\right)^{-\alpha}\right]^{\theta-1}}{1-\left[\frac{1-\left(\frac{t}{\beta}+1\right)^{-\alpha}}{1-\left(\frac{\beta}{1+\beta}\right)^{\alpha}}\right]^{\theta}} \quad (10)$$

$$\xi_3(t)_{TEL} = -ln[1-F(t)_{TEL}] = -ln\left[1-\left[\frac{1-\left(\frac{t}{\beta}+1\right)^{-\alpha}}{1-\left(\frac{\beta}{1+\beta}\right)^{\alpha}}\right]^{\theta}\right] \quad (11)$$

$$\xi_4(\theta)_{TEL} = \frac{f(t)_{TEL}}{F(t)_{TEL}} = \frac{\alpha\theta}{\beta}\left(1+\frac{t}{\beta}\right)^{-(\alpha+1)}\left[1-\left(1+\frac{t}{\beta}\right)^{-\alpha}\right]^{-1} \quad (12)$$

2.3 The r^{th} Moment of TEL Distribution

The r^{th} Moment of TEL Distribution can be derived as:

$$E(t^r)_{TEL} = \int_0^{\infty} t^r f(\theta)_{TEL} d\theta$$

$$= \frac{\alpha\theta}{\beta\left[1-\left(\frac{\beta}{1+\beta}\right)^{\alpha}\right]^{\theta}}\int_0^{\infty} t^r\left(\frac{t}{\beta}+1\right)^{-(\alpha+1)}\left[1-\left(\frac{t}{\beta}+1\right)^{-\alpha}\right]^{\theta-1} dt$$

Let $y = \left(\frac{t}{\beta}+1\right) \Rightarrow x = \beta(y-1) \Rightarrow dt = \beta dy,$ if $t \to 0 \Rightarrow y = 1$ and $t = 1 \Rightarrow y = \left(\frac{1+\beta}{\beta}\right)$ then

$$E(t^r)_{TEL} == \frac{\alpha\theta\beta^r}{\left[1-\left(\frac{\beta}{\beta+1}\right)^{\alpha}\right]^{\theta}} \int_{1}^{\frac{\beta+1}{\beta}} (-1)^r (1-y)^r\, y^{-(\alpha+1)} [1-y^{-\alpha}]^{\theta-1} dy \qquad (13)$$

Since, $(1-y)^r$ can be rewritten (see [8]) as

$$(1-y)^r = \sum_{k=0}^{\infty} (-1)^k \binom{r}{k} y^k \qquad (14)$$

Substituting (10) in (9) we get

$$E(t^r)_{TEL} = \frac{\alpha\theta\beta^r}{\left[1-\left(\frac{\beta}{\beta+1}\right)^{\alpha}\right]^{\theta}} \int_{1}^{\frac{\beta+1}{\beta}} \sum_{k=0}^{\infty} (-1)^{k+r} \binom{r}{k} y^{-(\alpha+1)+k} [1-y^{-\alpha}]^{\theta-1} dy \quad (15)$$

Furthermore, $[1-y^{-\alpha}]^{\theta-1}$ can be rewritten as (see [8])

$$[1-y^{-\alpha}]^{\theta-1} = \sum_{i=0}^{\infty} (-1)^i \binom{\theta-1}{i} y^{-i\alpha} \qquad (16)$$

Substituting (16) in (15) we get

$$E(t^r)_{TEL} = \alpha\theta\beta^r \int_{1}^{\frac{\beta+1}{\beta}} \sum_{k,i=0}^{\infty} (-1)^{k+r+i} \binom{r}{k} \binom{\theta-1}{i} y^{k-\alpha(i+1)-1} dy$$

$$= \alpha\theta\beta^r \sum_{k,i=0}^{\infty} (-1)^{k+r+i} \binom{r}{k} \binom{\theta-1}{i} \left(\frac{\left(\frac{\beta+1}{\beta}\right)^{k-\alpha(i+1)} - 1}{k-\alpha(i+1)} \right)$$

Then, $E(t^r)_{TEL}$, will be

$$E(t^r)_{TEL} = \alpha\theta\beta^r \sum_{k,i=0}^{\infty} (-1)^{k+r+i} \binom{r}{k} \binom{\theta-1}{i} \left(\frac{\left(\frac{\beta+1}{\beta}\right)^{k-\alpha(i+1)} - 1}{k-\alpha(i+1)} \right) \qquad (17)$$

2.4 The Characteristic Function of the TEL Distribution

Consequently, the "characteristic function" of the TEL distribution is given by,

$$\varphi_p(t)_{TEL} = E(e^{ipt})_{TEL} = \int_{0}^{\infty} e^{ipt} f(t)_{TEL} dt \qquad (18)$$

Since $e^{ipt} = \sum_{r=0}^{\infty} \frac{(ipt)^r}{r!}$, the $\varphi_p(t)_{TTEL}$ in (14) can be rewritten as (see [8])

$$\varphi_p(t)_{TTEL} = \int_0^1 \sum_{r=0}^{\infty} \frac{(ip)^r}{r!} f(t)_{TTEL} dt = \sum_{r=0}^{\infty} \frac{(ip)^r}{r!} E(t^r)_{TTEL} \qquad (19)$$

Where $E(t^r)_{TEL}$ as in (17).

2.5 Quantile Function

Through inverting the cdf in (7) the quantile function of the TEL distribution can be attained as follows

$$t_q = Q(t) = \beta \left[\frac{1 - \left(1 - u^{\frac{1}{\theta}}\left(1 - \left(\frac{\beta}{\beta+1}\right)^{\alpha}\right)\right)^{\frac{1}{\alpha}}}{\left(1 - u^{\frac{1}{\theta}}\left(1 - \left(\frac{\beta}{\beta+1}\right)^{\alpha}\right)\right)^{\frac{1}{\alpha}}} \right] \qquad (20)$$

The median of the TEL random variable can be gained from (20) by setting $q = 1/2$ as:

$$Median_{TEL} = Q(1/2) = \beta \left[\frac{1 - \left(1 - \left(\frac{1}{2}\right)^{\frac{1}{\theta}}\left(1 - \left(\frac{\beta}{\beta+1}\right)^{\alpha}\right)\right)^{\frac{1}{\alpha}}}{\left(1 - \left(\frac{1}{2}\right)^{\frac{1}{\theta}}\left(1 - \left(\frac{\beta}{\beta+1}\right)^{\alpha}\right)\right)^{\frac{1}{\alpha}}} \right] \qquad (21)$$

By replacing q with u, a random variable that follows TEL distribution can be simulated as:

$$t_{TEL} = Q(t) = \beta \left[\frac{1 - \left(1 - q^{\frac{1}{\theta}}\left(1 - \left(\frac{\beta}{\beta+1}\right)^{\alpha}\right)\right)^{\frac{1}{\alpha}}}{\left(1 - q^{\frac{1}{\theta}}\left(1 - \left(\frac{\beta}{\beta+1}\right)^{\alpha}\right)\right)^{\frac{1}{\alpha}}} \right] \qquad (22)$$

Where u is an interval-based uniform random number [0, 1].

2.6 Stress-Strength Relationship in the TEL Distribution

The two independent, different random variables U STRESS and V strength that follow TEL distribution, Then the stress-strength (SS_{TEL}) of the TEL distribution is expressed as follows, [8]:

$$SS_{TEL} = P(Y < Z)_{TEL} = \int_0^{\infty} F_U(t)_{TEL} f_V(t)_{TEL} dt \qquad (23)$$

Where $F_U(\theta)_{TEL}$ considers the cdf of the TEL distribution like in (7) with parameters β_1, α_1 and $\theta_1$as

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$F_U(t) = \frac{\left[1-\left(1+\frac{t}{\beta_1}\right)^{-\alpha_1}\right]^{\theta_1}}{\left[1-\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1}\right]^{\theta_1}}$ and $f_V(t)_{TEL}$ represents the pdf of TTEL distribution with parameters $\beta, \alpha,$ and θ as in (7). Then the stress-strength (SS_{TTEL}) can be rewritten as

$$SS_{TEL} = \int_0^1 \frac{\left[1-\left(1+\frac{t}{\beta_1}\right)^{-\alpha_1}\right]^{\theta_1}}{\left[1-\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1}\right]^{\theta_1}} f(\alpha,\beta;t)dt \qquad (24)$$

Based on the formula $(1-u)^b = \sum_{m=0}^{\infty}(-1)^m\binom{b}{m}u^m, |u|<1, b>0$ see [8], $\left[1-\left(1+\frac{t}{\beta_1}\right)^{-\alpha_1}\right]^{\theta_1} = \sum_{m=0}^{\infty}(-1)^m\binom{\theta_1}{m}\left(1+\frac{t}{\beta_1}\right)^{-\alpha_1 m}$ Now

$$SS_{TEL} = \frac{1}{\left[1-\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1}\right]^{\theta_1}} \int_0^1 \sum_{m=0}^{\infty}(-1)^m\binom{\theta_1}{m}\left(1+\frac{t}{\beta_1}\right)^{-\alpha_1 m} f(\alpha,\beta;t)dt$$

And so, Based on $(1-u)^{-b} = \sum_{i=0}^{\infty}\frac{\Gamma(b+i)}{\Gamma(b)}u^i, |u|<1, b>0$ (see [8] ,

$\left(1-\left(\frac{-t}{\beta_1}\right)\right)^{-\alpha_1 m} = \sum_{l=0}^{\infty}\frac{(-1)^l\Gamma(\alpha_1 m+l)}{\beta_1^l\Gamma(\alpha_1 m)}t^l$ we get

$$SS_{TEL} = \frac{1}{\left[1-\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1}\right]^{\theta_1}} \sum_{m,l=0}^{\infty}\frac{(-1)^{m+l}\Gamma(\alpha_1 m+l)}{\beta_1^l\Gamma(\alpha_1 m)}\binom{\theta_1}{m}\int_0^1 t^l f(\alpha,\beta;t)dt$$

$$= \frac{1}{\left[1-\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1}\right]^{\theta_1}} \sum_{m,l=0}^{\infty}\frac{(-1)^{m+l}\Gamma(\alpha_1 m+l)}{\beta_1^l\Gamma(\alpha_1 m)}\binom{\theta_1}{m}E(t^l)$$

Then, the stress-strength (SS_{TEL}) of the TEL distribution is given by:

$$SS_{TEL} = \frac{\alpha\theta}{\left[1-\left(\frac{\beta_1}{\beta_1+1}\right)^{\alpha_1}\right]^{\theta_1}} \sum_{m,l,k,i=0}^{\infty}\frac{(-1)^{k+2l+i+m}\beta^l\Gamma(\alpha_1 m+l)}{\beta_1^l\Gamma(\alpha_1 m)(\alpha(i+1)-k)}\binom{\theta_1}{m}\binom{\theta-1}{i}\binom{l}{k} \qquad (25)$$

2.7 Shannon Entropy of TEL Distribution

The Shannon entropy SH_{TEL} of TEL distribution can be obtained as [8] $E(-\ln f(t)_{TEL})$ Since

$\ln f(t)_{TEL} = \ln\left(\frac{\alpha\theta}{\beta}\right) - (\alpha+1)\ln\left(1+\frac{t}{\beta}\right) + (\theta-1)\ln\left[1-\left(1+\frac{t}{\beta}\right)^{-\alpha}\right] - \theta ln\left[1-\left(\frac{\beta}{\beta+1}\right)^{\alpha}\right]$, then

$$SH_{TEL} = -\ln\left(\frac{\alpha\theta}{\beta}\right) + (\alpha+1)E\left(\ln\left(1+\frac{t}{\beta}\right)\right) - (\theta-1)E\left(\ln\left[1-\left(1+\frac{t}{\beta}\right)^{-\alpha}\right]\right) + \theta ln\left[1-\left(\frac{\beta}{\beta+1}\right)^{\alpha}\right] \qquad (26)$$

Let, $I_1 = E\left(\ln\left(1+\frac{t}{\beta}\right)\right)$, $I_2 = E\left(\ln\left[1-\left(1+\frac{t}{\beta}\right)^{-\alpha}\right]\right)$

Now fore, $I_1 = \int_0^\infty \ln\left(1+\frac{t}{\beta}\right) f(t)_{TEL} dt$

Based on $\ln(1-u) = -\sum_{j=0}^{\infty} \frac{1}{j} u^j, |u| < 1$ *see* [8],

$\ln\left(1-\left(\frac{-t}{\beta}\right)\right) = \sum_{j=0}^{\infty} \frac{(-1)^{j+1}}{j\beta^j} t^j$ Now

$$I_1 = \sum_{j=0}^{\infty} \frac{(-1)^{j+1}}{j\beta^j} \int_0^\infty t^j f(t)_{TEL} d = \sum_{j=0}^{\infty} \frac{(-1)^{j+1}}{j\beta^j} E(t^j)$$

Based on (17), with r=j we get

$$I_1 = \sum_{k,i,j=0}^{\infty} \alpha\theta \frac{(-1)^{k+2j+i+1}}{j(\alpha(i+1)-k)} \binom{j}{k} \binom{\theta-1}{i} \quad (27)$$

For $I_2 = E\left(\ln\left[1-\left(1+\frac{t}{\beta}\right)^{-\alpha}\right]\right) = \int_0^\infty \ln\left[1-\left(1+\frac{t}{\beta}\right)^{-\alpha}\right] f(t)_{TEL} dt$

Using on previous formula, we get

$\ln\left[1-\left(1+\frac{t}{\beta}\right)^{-\alpha}\right] = -\sum_{h=0}^{\infty} \frac{1}{h}\left(1+\frac{t}{\beta}\right)^{-\alpha h} = \sum_{l,h=0}^{\infty} \frac{(-1)^{l+1}\Gamma(\alpha m+l)}{\beta^l \Gamma(\alpha m)} t^l$ Now

$$I_2 = \sum_{l,h=0}^{\infty} \frac{(-1)^{l+1}\Gamma(\alpha m+l)}{\beta^l \Gamma(\alpha m)} \int_0^\infty t^l f(t)_{TEL} dt = \sum_{l,h=0}^{\infty} \frac{(-1)^{l+1}\Gamma(\alpha m+l)}{\beta^l \Gamma(\alpha m)} E(t^l) \quad (28)$$

Based on (28) and (13), with $r = l$ we get

$$I_2 = \sum_{l,h,k,i=0}^{\infty} \alpha\theta \frac{(-1)^{k+2l+i+1}\Gamma(\alpha m+l)}{\Gamma(\alpha m)\,(\alpha(i+1)-k)} \binom{l}{k} \binom{\theta-1}{i} \quad (29)$$

Therefore, Shannon entropy of TEL distribution can be obtained as

$$SH_{TEL} = -\ln\left(\frac{\alpha\theta}{\beta}\right) + (\alpha+1)I_1 - (\theta-1)I_2 + \theta ln\left[1-\left(\frac{\beta}{\beta+1}\right)^{\alpha}\right] \quad (30)$$

3. Maximum Likelihood Estimator method (MLE)

Let $t = (t_1, t_2, \dots, t_n$ be a random sample (r.s.) of size (n) of the TEL distribution, the likelihood function is: $L(\alpha,\beta;t) = \prod_{i=1}^{n} f(t_i;\alpha,\beta)$

$$L(\alpha,\beta;t_i) = \frac{\frac{\alpha^n\theta^n}{\beta^n}\prod_{i=1}^{n}\left(1+\frac{t_i}{\beta}\right)^{-(\alpha+1)} \prod_{i=1}^{n}\left[1-\left(1+\frac{t_i}{\beta}\right)^{-\alpha}\right]^{\theta-1}}{\left[1-\left(\frac{\beta}{\beta+1}\right)^{\alpha}\right]^{n\theta}} \quad (31)$$

Then, with regard to β,α, and θ and equating to zero, the partial derivative for the Log Likelihood function is carried out. The MLE estimation of β,α, and θ is indicated by β ̂_MLE,α ̂_MLE, and θ ̂_MLE, and they are as follows:

$$\hat{\beta}_{\text{MLE}} = -\frac{n}{\beta} + (\alpha+1)\sum_{i=1}^{n}\frac{\frac{t_i}{\beta^2}}{\left(1+\frac{t_i}{\beta}\right)} - \sum_{i=1}^{n}\frac{\alpha(\theta-1)\frac{t_i}{\beta^2}}{\left[1-\left(1+\frac{t_i}{\beta}\right)^{-\alpha}\right]\left(1+\frac{t_i}{\beta}\right)^{\alpha+1}} - \frac{n\theta\alpha\left(\frac{\beta}{\beta+1}\right)^{\alpha-1}}{\left[1-\left(\frac{\beta}{\beta+1}\right)^{\alpha}\right](\beta+1)^2} \quad (32)$$

$$\hat{\alpha}_{\text{MLE}} = \frac{n}{\alpha} - \ln\left(1+\frac{t_i}{\beta}\right) + (\theta-1)\sum_{i=1}^{n}\frac{\ln\left(1+\frac{t_i}{\beta}\right)}{\left(1+\frac{t_i}{\beta}\right)^{\alpha}\left[1-\left(1+\frac{t_i}{\beta}\right)^{-\alpha}\right]} - \frac{n\alpha\left(\frac{\beta}{\beta+1}\right)^{\alpha-1}\ln\left(\frac{\beta}{\beta+1}\right)}{\left[1-\left(\frac{\beta}{\beta+1}\right)^{\alpha}\right]} \quad (33)$$

$$\hat{\theta}_{\text{MLE}} = \frac{n}{\theta} + \sum_{i=1}^{n}\ln\left[1-\left(1+\frac{t_i}{\beta}\right)^{-\alpha}\right] - n\ln\left[1-\left(\frac{\beta}{\beta+1}\right)^{\alpha}\right] \quad (34)$$

4. Simulation study

The simulation study, In this section, was performed as empirical method to determine the behavior of the $MLEs$ parameters of the TEL for sample sizes (n = 25, 50, 100), and using the formulas in (31), (32) and (33) with the default parameter values shown in Table (1,2, …, 8) with α = 0.5, β = 0.2, 2 and$\theta = 0.5, 2$. The mean square error (MSE) was used as a criterion for comparisons and evaluations. Using R codes, we obtained the results.

Table -1 empirical estimates (MSE) of the TEL distribution where the parameters:α = 0.5, β = 0.5, θ = 0.5

Parameters		*α = 0.5*	*β = 0.5*	*θ = 0.5*
n=25	Estimates	0.51319	0.48114	0.53496
	MSE	***0.03513***	***0.03286***	***0.01249***
n=50	Estimates	0.50468	0.49098	0.51489
	MSE	***0.01847***	***0.01667***	***0.00549***
n=100	Estimates	0.50113	0.49694	0.50908
	MSE	***0.00910***	***0.00843***	***0.00277***

Table -2 empirical estimates (MSE) of the TEL distribution where the parameters:α = 0.5, β = 0.5, θ = 2

Parameters		*α = 0.5*	*β = 0.5*	*θ = 2*
n=25	Estimates	0.51920	0.47132	2.24629
	MSE	***0.03680***	***0.03319***	***0.29243***
n=50	Estimates	0.51268	0.48609	2.11960
	MSE	***0.01775***	***0.01672***	***0.13754***
n=100	Estimates	0.50892	0.48958	2.05865
	MSE	***0.00913***	***0.00845***	***0.06113***

Table -3 empirical estimates (MSE) of the TEL distribution where the parameters: $\alpha = 0.5$, $\beta = 2$, $\theta = 0.5$

Parameters		$\alpha = 0.5$	$\beta = 2$	$\theta = 0.5$
n=25	Estimates	0.51546	1.91999	0.52028
	MSE	***0.03847***	***0.58022***	***0.01040***
n=50	Estimates	0.50905	1.95391	0.51180
	MSE	***0.01930***	***0.29566***	***0.00491***
n=100	Estimates	0.50465	1.97026	0.50863
	MSE	***0.00957***	***0.14528***	***0.00259***

Table -4 empirical estimates (MSE) of the TEL distribution where the parameters: $\alpha = 0.5$, $\beta = 2$, $\theta = 2$

Parameters		$\alpha = 0.5$	$\beta = 2$	$\theta = 2$
n=25	Estimates	0.52446	1.87437	2.17190
	MSE	***0.03834***	***0.58963***	***0.20332***
n=50	Estimates	0.50969	1.95676	2.06652
	MSE	***0.01926***	***0.29269***	***0.08662***
n=100	Estimates	0.50882	1.95537	2.05027
	MSE	***0.00965***	***0.14615***	***0.04525***

Table -5 empirical estimates (MSE) of the TEL distribution where the parameters: $\alpha = 2$, $\beta = 0.5$, $\theta = 0.5$

Parameters		$\alpha = 2$	$\beta = 0.5$	$\theta = 0.5$
n=25	Estimates	2.03554	0.48756	0.53969
	MSE	***0.45615***	***0.02938***	***0.01401***
n=50	Estimates	2.02135	0.49498	0.51957
	MSE	***0.23108***	***0.01417***	***0.00664***
n=100	Estimates	2.00595	0.49618	0.50752
	MSE	***0.11483***	***0.00735***	***0.00318***

Table -6 empirical estimates (MSE) of the TEL distribution where the parameters: $\alpha = 2$, $\beta = 0.5$, $\theta = 2$

Parameters		$\alpha = 2$	$\beta = 0.5$	$\theta = 2$
n=25	Estimates	2.04466	0.49197	2.23274
	MSE	***0.42535***	***0.02559***	***0.33679***
n=50	Estimates	1.99560	0.50129	2.08534
	MSE	***0.22393***	***0.01343***	***0.16122***
n=100	Estimates	2.01921	0.49747	2.06502
	MSE	***0.10621***	***0.00667***	***0.08385***

Table -7 empirical estimates (MSE) of the TEL distribution where the parameters: $\alpha = 2$, $\beta = 2$, $\theta = 0.5$

Parameters		$\alpha = 2$	$\beta = 2$	$\theta = 0.5$
n=25	Estimates	2.00473	1.90830	0.53943
	MSE	***0.54401***	***0.53347***	***0.01246***
n=50	Estimates	2.01484	1.97544	0.52079
	MSE	***0.27097***	***0.26614***	***0.00579***
n=100	Estimates	2.01493	1.99791	0.51061
	MSE	***0.13788***	***0.13410***	***0.00280***

Table -8 empirical estimates (MSE) of the TEL distribution where the parameters: $\alpha = 2$, $\beta = 2$, $\theta = 2$

Parameters		$\alpha = 2$	$\beta = 2$	$\theta = 2$
n=25	Estimates	2.08280	1.92810	2.25727
	MSE	***0.53149***	***0.49443***	***0.30380***
n=50	Estimates	1.99699	1.96748	2.11546
	MSE	***0.27106***	***0.25456***	***0.13555***
n=100	Estimates	2.02748	1.98755	2.06883
	MSE	***0.13785***	***0.12964***	***0.07073***

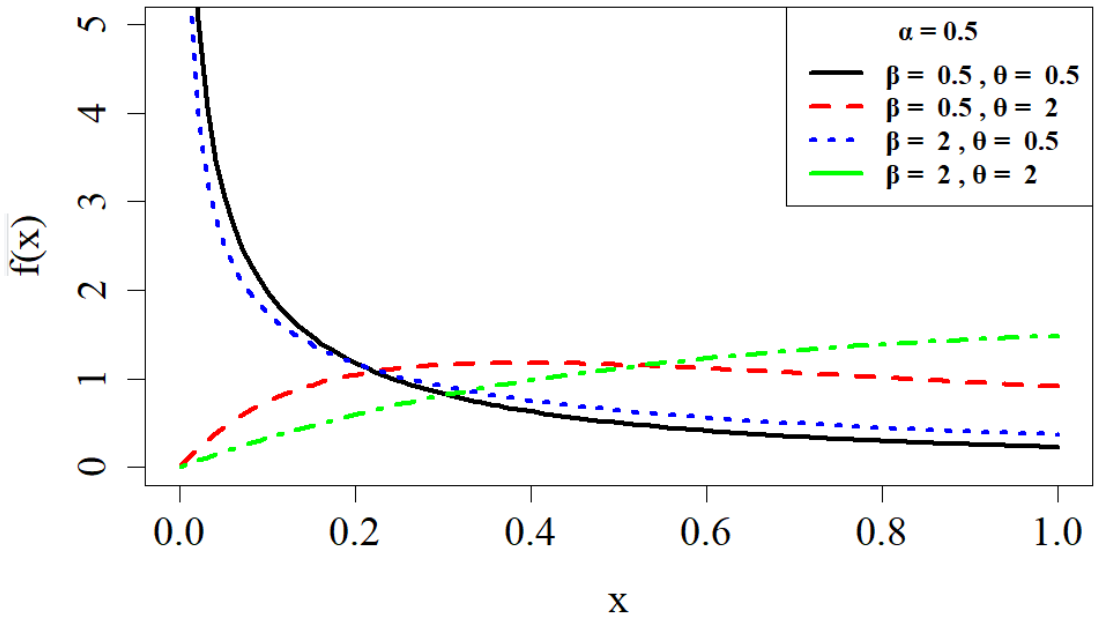

Figure -1 Depict the graphical representation of the probability density function (PDF) of the TEL distribution with varying parameter values β=0.5, 2, $\alpha = 0.5$ and θ=0.5, 2.

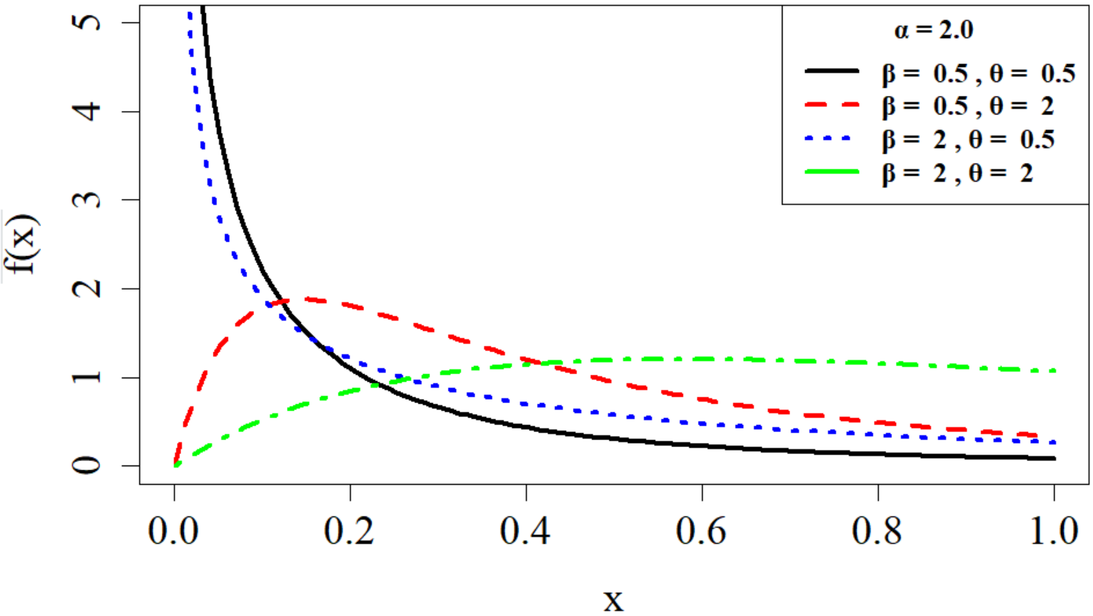

Figure -2 Depict the graphical representation of the probability density function (PDF) of the TEL distribution with varying parameter values β=0.5, 2, $\alpha = 2$ and θ=0.5, 2.

For all tables, From the empirical results we can see the following:

- ❖ As shown for all the tables, we see that the MSE is more accurate when the default values for all parameters are small for all sample sizes.
- ❖ We notice that there is a direct proportion between the worths of the parameters α, β and the values of the MSE, that is, when the parameter θ or β is increased, the value of MSE increases.
- ❖ In general, the values associated with the parameters decrease with increasing sample size. This result is consistent with the statistics theory.

5. Concluding Remarks

In this research, the Lomax distribution was limited to the period [0, 1] and we noticed that the transformed distribution (TEL) is more flexible than the original distribution. Also, the most important statistical properties were found and derived, represented by the rth moment, characteristic function, quantile function, reliability measures, stress-strength along with Shannon entropy Also, by comparing the default values of the maximum possibility estimator and the mean square error, it was revealed that there is a direct proportionality between the two concepts.

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