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A New Method to Solve Second-Order Boundary Value Problems by Exponential Spline Functions

Ahmed R Khlefha

[1]*Department of Mathematics, College of Education, University of Sumer, Thi-Qar, Iraq*.

*E-mail of corresponding author: arkdsh85@gmail.com

Abstract

In this research, linear and nonlinear second-order boundary value problems are solved, by using exponential spline functions. The suggested method's validity and applicability are proved by a few numerical findings. The method's results show that it is both easy to use and efficient.

Keywords: Second-order boundary value problem, exponential spline method, exact solution, Absolute errors, Finite difference method.

طريقة جديدة لحل مسائل القيم المحددة من الدرجة الثانية بواسطة دوال الثلمة الاسية

احمد رشيد خليفة

قسم الرياضيات,كلية التربية,جامعة سومر ,ذي قار , العراق

الخلاصة

في هذا البحث، تم حل مسائل القيم الحدودية الخطية والغير خطية من الدرجة الثانية، باستخدام الدوال الثلمه الأسية. تم إثبات صحة الطريقة المقترحة وقابليتها للتطبيق من خلال بعض النتائج العددية. التي تظهر نتائج الطريقة أنها سهلة الاستخدام وفعالة

1. Introduction

A wide range of issues in the fields of research, technology, and mathematics can be simplified by resolving systems of equations. This is particularly relevant in tasks such as modelling and simulating physical systems and verifying and validating engineering designs [1]. Those boundary value problems (BVPs) using various forms of boundary conditions (BCs) are effective tools for defining several realistic situations and hence, represent a highly engaging subject for scholars. The application of mathematical models to represent real-world problems as systems of boundary value problems (BVPs) is commonly observed in various fields Topics include dynamics of populations, brine container process, compartments analysis, pricing of a drug called lid flow of nutrients in aquariums, trains, electrical networks, chemo facts, racing hearts, coupled spring-mass structures, timber recording by plane, shaking effects on properties, aquatic contamination, etc [2]. Many scholars have recently focused on analyzing second-order systems of boundary value problems (BVPs). Considerable work has been dedicated to resolving those challenges using numerical means, resulting in the development of effective and precise methodologies. Geng and Cui [3] examined the second-order linear and nonlinear systems in the replicating kernel space. Ogunlaran et al. [4] employed their method to resolve the system of non-linear divergent equations, while Gamel. [5] and Dehghan et al [6] introduced a sinc-collocation approach to resolve these systems. The treatment of boundary value problems (BVPs) by employing spline function is a subject of ongoing research due to its wide variety of implementations and mathematical implications [7]. Spline function-based techniques have effectively simulated them to sets of boundary value problems (BVPs). Heilat et al. [8] utilized an expanded technique based on cubic B-splines and resolved the linear instance of the aforementioned problem. Goh and his colleagues [9] successfully resolved singularity boundary value problems using an improved regular B-spline polynomials-dependent technique. We consider a non-linear system of second-order boundary value problems of the form [10], [11]:

$$\left.\begin{array}{l}\hat{y}^{(2)}(\ddot{\eta})+q_1(\ddot{\eta})\hat{y}^{(1)}(\ddot{\eta})+q_2(\ddot{\eta})\hat{y}(\ddot{\eta})+q_3(\ddot{\eta})\widehat{w}^{(1)}(\ddot{\eta})+q_4(\ddot{\eta})\widehat{w}(\ddot{\eta})+S_1(\ddot{\eta},\hat{y},\widehat{w})=r_1(\ddot{\eta})\\ \widehat{w}^{(2)}(\ddot{\eta})+p_1(\ddot{\eta})\widehat{w}^{(1)}(\ddot{\eta})+p_2(\ddot{\eta})\widehat{w}(\ddot{\eta})+p_3(\ddot{\eta})\hat{y}^{(1)}(\ddot{\eta})+p_4(\ddot{\eta})y(\ddot{\eta})+S_2(\ddot{\eta},\hat{y},\widehat{w})=r_2(\ddot{\eta})\end{array}\right\}. \quad (1)$$

under the given conditions

$$\hat{y}(0)=\hat{y}(1)=0, \widehat{w}(0)=\widehat{w}(0)=0. \quad (2)$$

Given functions $r_1(\ddot{\eta})$ and $r_2(\ddot{\eta})$, where$0<\ddot{\eta}<1$, S_1 , S_2 are non-linear functions of$\hat{y}$ and $\widehat{w}$. are given functions, and$q_i(\ddot{\eta}), p_i(\ddot{\eta})$are continuous, $i=1,2,3,4$. It is important to note that the system of second-order boundary value problems (BVPs) we are discussing is simply a specific instance of the problem (1)-(2) that we expect to encounter. The comprehensive elucidation on the presence of solutions for these systems may be readily located in [12], [13] . It is widely believed that within the required time frame, the suggested system (1)-(2) has just one solution. It is evident that many methods such as replicating kernel, sinc-collocation, and variational iteration have been proposed to find the solution to the second-order boundary value problems, for both linear and non-linear instances. In this study, we present a highly efficient numerical technique based on non-polynomial cubic splines to solve the given system of equations (1)-(2). Our solution technique relies upon the utilization of an approach composed of exponentially spline functions. This approach effectively resolves the purposeful problem at hand. The current technique is formulated using the subsequent function space:

$$T_3(\ddot{\eta})=Span\left(1,\ddot{\eta},e^{\widetilde{\omega}\ddot{\eta})}\right)= Span\left(1,\ddot{\eta},\left(e^{\widetilde{\omega}\ddot{\eta})}-\widetilde{\omega}\ddot{\eta}\right)\right)$$

Here, $\widetilde{\omega}$ represents the proportion of the non-polynomial functions. If the limit of $\widetilde{\omega}$ approaches 0, then $T_3(\ddot{\eta})$ can be simplified to the span of $\{1,\ddot{\eta},\ddot{\eta}^2,\ddot{\eta}^3\}$ [14]. Chaurasia et al. [15] have examined the current configuration of function space to address the fourth-order system of boundary value problems, albeit using quintic non-polynomial splines.

Our work has been structured in this manner. In section 2, we have discussed the implementation of exponential spline technique. The third portion of the document provides a comprehensive approach to solving a second-order system of boundary value problem. Section three includes the resolution of two examples to confirm the feasibility of our devised approach, accompanied by graphical representations. The study is concluded in Section seven

2-Exponential Spline method

The interval [a, b] is partitioned into n subintervals of equal length by introducing the point $x_i = a + ih, i = 0,1,2, \ldots, n,$where takes on values from 0 to n. with $a = \ddot{\eta}_0,\ b = \ddot{\eta}_n$ and $h = \frac{b-a}{n}$, where n is an arbitrary positive integer.

Let $\hat{y}(\ddot{\eta})$ represent the precise solution, and let $\hat{y}(\ddot{\eta}_i)$ be an approximation obtained through the use of exponential spline $E_i(\ddot{\eta})$. This spline is constructed to pass through the points $(\ddot{\eta}_i, \hat{y}_i)$ and $(\ddot{\eta}_{i+1}, y_{i+1}$. In addition to satisfying the interpolator condition at $\ddot{\eta}_i$ and $\ddot{\eta}_{i+1}$,it is also required that the first derivative of $E_i(\ddot{\eta})$.is continuous at the shared nodes $(\ddot{\eta}_i, \hat{y}_i)$. The expression $E_i(\ddot{\eta})$. Is written in the following form:

$$E_i(x) = a_i e^{\widetilde{\omega}(\ddot{\eta}-\ddot{\eta}_i)} + b_i e^{-\widetilde{\omega}(\ddot{\eta}-\ddot{\eta}_i)} + c_i(\ddot{\eta} - \ddot{\eta}_i) + d_i. \tag{3}$$

In the context of function interpolation, exponential spline function $E(\ddot{\eta})$ belonging to class $C^2[a, b]$is utilized to approximate the values of $\hat{y}(\ddot{\eta})$ at various points $\ddot{\eta}_i, i = 0,1,2, \ldots, n$. This approximation is influenced by a factor $\widetilde{\omega}$ and tends towards the ordinary spline $E(\ddot{\eta})$ inside the interval $[a, b]$ as the parameter $\widetilde{\omega}$ approaches zero.

In order to obtain the formula for the coefficients of the equation (2), it is necessary to perform a derivation $\hat{y}_i, \hat{y}_{i+1}, F_i$ and F_{i+1}we first define [3]:

$$E_i(\ddot{\eta}_i) = \hat{y}_i\ ,\ E_i(\ddot{\eta}_{i+1}) = \hat{y}_{i+1}\ ,\ E_i^{(2)}(\ddot{\eta}_i) = F_i\ ,\ E_i^{(2)}(\ddot{\eta}_{i+1}) = F_{i+1}. \tag{4}$$

Through the process of algebraically manipulating, the next statement is obtained.

$$\begin{cases} a_i = \frac{h^2(-F_i e^{-\theta} + F_{i+1}),}{\theta^2(e^{\theta} - e^{-\theta})} \\ b_i = \frac{h^2(F_i e^{\theta} - F_{i+1})}{\theta^2(e^{\theta} - e^{-\theta})}, \\ c_i = \frac{-h(F_{i+1} - k^2\hat{y}_{i+1} + k^2\hat{y}_i - F_i)}{\theta^2}, \\ d_i = \frac{(\theta^2\hat{y}_i - h^2F_i)}{\theta^2}. \end{cases} \tag{5}$$

Where $\theta = \widetilde{\omega}h$ and $i = 0,1,2, \ldots, n$.

We will apply the first derivative $(\ddot{\eta}_i, \hat{y}_i)$, that is $E_{i-1}^{(1)}(\ddot{\eta}_i) = E_i^{(1)}(\ddot{\eta}_i)$,gives the following for $i = 1, \ldots, n$:

$$\hat{y}_{i-1} - 2\hat{y}_i + \hat{y}_{i+1} = \left(\frac{(\theta e^{-\theta} - 2\theta e^{\theta} + e^{2\theta} + e^{-2\theta} - 2)h^2}{\theta^2(e^{2\theta} + e^{-2\theta} - 2)}\right) F_{i-1} + 2\left(\frac{(\theta e^{2\theta} - \theta e^{-2\theta} - e^{2\theta} - e^{-2\theta} + 2)h^2}{\theta^2(e^{2\theta} + e^{-2\theta} - 2)}\right) F_i + \left(\frac{(\theta e^{-\theta} - 2\theta e^{\theta} + e^{2\theta} + e^{-2\theta} - 2)h^2}{\theta^2(e^{2\theta} + e^{-2\theta} - 2)}\right), \tag{6}$$

Which we can write as follows,

$$(\hat{y}_{i-1} - 2\hat{y}_i + \hat{y}_{i+1}) = h^2[\mu(F_{i-1} + F_{i+1}) + 2\lambda F_i]\ , \tag{7}$$

$w(\ddot{\eta})$ is written in a similar fashion.

$$(\widehat{w}_{i-1} - 2\widehat{w}_i + \widehat{w}_{i+1}) = h^2[\mu(G_{i-1} + G_{i+1}) + 2\lambda G_i] \ , \tag{8}$$

Where,

$$\mu = \frac{(\theta e^{-\theta} - 2\theta e^{\theta} + e^{2\theta} + e^{-2\theta} - 2)}{\theta^2(e^{2\theta} + e^{-2\theta} - 2)}, \lambda = \frac{(\theta e^{2\theta} - \theta e^{-2\theta} - e^{2\theta} - e^{-2\theta} + 2)}{\theta^2(e^{2\theta} + e^{-2\theta} - 2)}.$$

3-Application to the second order boundary value problems

In order to demonstrate the practical use of the exponential spline method discussed earlier, we will examine a nonlinear second-order boundary value problems, as defined in Equation (1). The suggested second-order boundary value problem in Equation (1) can be discretized at the grid point $(\ddot{\eta}_i, y_i)$.

$$\left.\begin{aligned}\hat{y}^{(2)}(\ddot{\eta})+q_1(\ddot{\eta})\hat{y}^{(1)}(\ddot{\eta})+q_2(\ddot{\eta})\hat{y}(\ddot{\eta})+q_3(\ddot{\eta})\widehat{w}^{(1)}(\ddot{\eta})+q_4(\ddot{\eta})\widehat{w}(\ddot{\eta})+S_1(\ddot{\eta},\hat{y},\widehat{w})=r_1(\ddot{\eta})\\ \widehat{w}^{(2)}(\ddot{\eta})+p_1(\ddot{\eta})\widehat{w}^{(1)}(\ddot{\eta})+p_2(\ddot{\eta})\widehat{w}(\ddot{\eta})+p_3(\ddot{\eta})\hat{y}^{(1)}(\ddot{\eta})+p_4(\ddot{\eta})\hat{y}(\ddot{\eta})+S_2(\ddot{\eta},\hat{y},\widehat{w})=r_2(\ddot{\eta})\end{aligned}\right\} \tag{9}$$

Substituting $F_i = \hat{y}^{(2)}(x)$ and $G_i = w^{(2)}(\ddot{\eta})$ in Equation system (5)

$$\left.\begin{aligned}F_i+q_1(\ddot{\eta})\hat{y}_i^{(1)}+q_2(\ddot{\eta})\hat{y}_i+q_3(\ddot{\eta})\widehat{w}_i^{(1)}+q_4(\ddot{\eta})\widehat{w}_i+S_1(\ddot{\eta},\hat{y},\widehat{w})=r_1(\ddot{\eta})\\ G_i+p_1(\ddot{\eta})w_i^{(1)}(\ddot{\eta})+p_2(\ddot{\eta})w_i+p_3(\ddot{\eta})\hat{y}_i^{(1)}+p_4(\ddot{\eta})\hat{y}_i+S_2(\ddot{\eta},\hat{y},\widehat{w})=r_2(\ddot{\eta})\end{aligned}\right\} \tag{10}$$

Solving equation (10) for F_i and G_i we get

$$\left.\begin{aligned}F_i=-q_1(\ddot{\eta})\hat{y}_i^{(1)}-q_2(\ddot{\eta})\hat{y}_i-q_3(\ddot{\eta})\widehat{w}_i^{(1)}-q_4(\ddot{\eta})\widehat{w}_i-S_1(\ddot{\eta},\hat{y},\widehat{w})+r_1(\ddot{\eta})\\ G_i=-p_1(\ddot{\eta})\widehat{w}_i^{(1)}-p_2(\ddot{\eta})\widehat{w}_i-p_3(\ddot{\eta})\hat{y}_i^{(1)}-p_4(\ddot{\eta})\hat{y}_i-S_2(\ddot{\eta},\hat{y},\widehat{w})+r_2(\ddot{\eta})\end{aligned}\right\} \tag{11}$$

The following approximations for the first order derivative of $\hat{y}$ and $\widehat{w}$ in Equation (11) can be using

$$\begin{cases} y'_{i-1}=\frac{-\hat{y}_{i+1}+4\hat{y}_i-3\hat{y}_{i-1}}{2h}\\ y'_i=\frac{\hat{y}_{i+1}-\hat{y}_{i-1}}{2h}\\ y'_{i+1}=\frac{3\hat{y}_{i+1}-4\hat{y}_i+\hat{y}_{i-1}}{2h}\\ \widehat{w}'_{i-1}=\frac{-\widehat{w}_{i+1}+4\widehat{w}_i-3\widehat{w}_{i-1}}{2h}\\ w'_i=\frac{\widehat{w}_{i+1}-\widehat{w}_{i-1}}{2h}\\ \widehat{w}'_{i+1}=\frac{3\widehat{w}_{i+1}-4\widehat{w}_i+\widehat{w}_{i-1}}{2h}\end{cases} \tag{12}$$

So, Equation (12) becomes

$$F_i = -q_1(\ddot{\eta})\frac{\hat{y}_{i+1}-\hat{y}_{i-1}}{2h} - q_2(\ddot{\eta})\hat{y}(\ddot{\eta}) - q_3(\ddot{\eta})\frac{\widehat{w}_{i+1}-\widehat{w}_{i-1}}{2h} - q_4(\ddot{\eta})\widehat{w}(\ddot{\eta}) - S_1(\ddot{\eta},\hat{y},\widehat{w}) + r_1(\ddot{\eta}) \tag{13}$$

$$F_{i+1} = -q_1(\ddot{\eta}_{i+1})\frac{3\hat{y}_{i+1}-4\hat{y}_i+\hat{y}_{i-1}}{2h} - q_2(\ddot{\eta}_{i+1})\hat{y}_{i-} - q_3(\ddot{\eta}_{i+1})\frac{3\widehat{w}_{i+1}-4\widehat{w}_i+\widehat{w}_{i-1}}{2h} - q_4(\ddot{\eta}_{i+1})\widehat{w}_i - S_1(\ddot{\eta}_{i+1},\hat{y}_{i+1},\widehat{w}_{i+1}) + r_1(\ddot{\eta}_{i+1}) \tag{14}$$

$$F_{i-1} = -q_1(\ddot{\eta}_{i-1})\frac{-\hat{y}_{i+1}+4\hat{y}_i-3\hat{y}_{i-1}}{2h} - q_2(\ddot{\eta}_{i-1})\hat{y}_{i-1} - q_3(\ddot{\eta}_{i-1})\frac{-\widehat{w}_{i+1}+4\widehat{w}_i-3\widehat{w}_{i-1}}{2h} - q_4(\ddot{\eta}_{i-1})\widehat{w}_i - S_1(\ddot{\eta}_{i+1},\hat{y}_{i+1},\widehat{w}_{i+1}) + r_1(\ddot{\eta}_{i-1}) \tag{15}$$

and

$$G_i = -p_1(\ddot{\eta}_i)\frac{\widehat{w}_{i+1}-\widehat{w}_{i-1}}{2h} - p_2(\ddot{\eta}_i)\widehat{w}_i - p_3(\ddot{\eta}_i)\frac{\hat{y}_{i+1}-\hat{y}_{i-1}}{2h} - p_4(\ddot{\eta}_i)\hat{y}_i - S_2(\ddot{\eta}_i,\hat{y}_i,\widehat{w}_i) + r_2(\ddot{\eta}_i) \quad (16)$$

$$G_{i+1} = -p_1(\ddot{\eta}_{i+1})\frac{3\widehat{w}_{i+1}-4\widehat{w}_i+\widehat{w}_{i-1}}{2h} - p_2(\ddot{\eta}_{i+1})\widehat{w}_i - p_3(\ddot{\eta}_{i+1})\frac{3\hat{y}_{i+1}-4\hat{y}_i+\hat{y}_{i-1}}{2h} - p_4(\ddot{\eta}_{i+1})\hat{y}_i - S_2(\ddot{\eta}_{i+1},\hat{y}_{i+1},\widehat{w}_{i+1}) + r_2(\ddot{\eta}_{i+1}) \quad (17)$$

$$G_{i-1} = -p_1(\ddot{\eta}_{i-1})\frac{-\widehat{w}_{i+1}+4\widehat{w}_i-3\widehat{w}_{i-1}}{2h} - p_2(\ddot{\eta}_{i-1})\widehat{w}_i - p_3(\ddot{\eta}_{i-1})\frac{-\hat{y}_{i+1}+4\hat{y}_i-3\hat{y}_{i-1}}{2h} - p_4(\ddot{\eta}_{i-1})\hat{y}_i - S_2(\ddot{\eta}_{i-1},\hat{y}_{i-1},\widehat{w}_{i-1}) + r_2(\ddot{\eta}_{i-1}) \quad (18)$$

Substituting equations. (13-15) - (16-18) in equations. (7) and (8) respectively, we find the following $2(n-1)$ linear algebraic equations in the $2(n+1)$unknowns for $n = 0,1 \dots, n$

$$\left[\frac{3\mu q_1(\ddot{\eta}_{i-1})}{2h} - \mu q_2(\ddot{\eta}_{i-1}) + \frac{2\lambda q_1(\ddot{\eta}_i)}{2h} - \frac{\mu q_1(\ddot{\eta}_{i+1})}{2h} - \frac{1}{h^2}\right]\hat{y}_{i-1} + \left[\frac{-4\mu q_1(\ddot{\eta}_{i-1})}{2h} - 2\mu q_2(\ddot{\eta}_i) + \frac{4\mu q_1(\ddot{\eta}_{i+1})}{2h} + \frac{2}{h^2}\right]\hat{y}_i + \left[\frac{\mu q_1(\ddot{\eta}_{i-1})}{2h} - \frac{2\lambda q_1(\ddot{\eta}_i)}{2h} - \frac{3\mu q_1(\ddot{\eta}_{i+1})}{2h} - \mu q_2(\ddot{\eta}_{i+1}) - \frac{1}{h^2}\right]\hat{y}_{i+1} + \left[\frac{3\mu q_3(\ddot{\eta}_{i-1})}{2h} - \mu q_4(\ddot{\eta}_{i-1}) + \frac{2\lambda q_3(\ddot{\eta}_i)}{2h} - \frac{\mu q_3(\ddot{\eta}_{i+1})}{2h}\right]\widehat{w}_{i-1} + \left[\frac{-4\mu q_3(\ddot{\eta}_{i-1})}{2h} - 2\lambda q_4(x_i) + \frac{4\mu q_3(\ddot{\eta}_{i+1})}{2h}\right]\widehat{w}_i + \left[\frac{\mu q_3(\ddot{\eta}_{i-1})}{2h} - \frac{2\lambda q_3(\ddot{\eta}_i)}{2h} - \frac{3\mu q_3(\ddot{\eta}_{i+1})}{2h} - \mu q_4(x_{i+1})\right]\widehat{w}_{i+1} = -\mu S_1(\ddot{\eta}_{i-1},y_{i-1},\widehat{w}_{i-1}) - 2\lambda S_1(\ddot{\eta}_i,\hat{y}_i,\widehat{w}_i) - \mu S_1(\ddot{\eta}_{i+1},\hat{y}_{i+1},\widehat{w}_{i+1}) - \mu r_1(\ddot{\eta}_{i-1}) + 2\lambda r_1(\ddot{\eta}_i) - \mu r_1(x_{i+1}) \quad (19)$$

And,

$$\left[\frac{3\mu p_1(\ddot{\eta}_{i-1})}{2h} - \mu p_2(\ddot{\eta}_{i-1}) + \frac{2\lambda p_1(\ddot{\eta}_i)}{2h} - \frac{\mu p_1(\ddot{\eta}_{i+1})}{2h} - \frac{1}{h^2}\right]\widehat{w}_{i-1} + \left[\frac{-4\mu p_1(\ddot{\eta}_{i-1})}{2h} - 2\mu p_2(\ddot{\eta}_i) + \frac{4\mu p_1(\ddot{\eta}_{i+1})}{2h} + \frac{2}{h^2}\right]\widehat{w}_i + \left[\frac{\mu p_1(\ddot{\eta}_{i-1})}{2h} - \frac{2\lambda p_1(\ddot{\eta}_i)}{2h} - \frac{3\mu p_1(\ddot{\eta}_{i+1})}{2h} - \mu p_2(\ddot{\eta}_{i+1}) - \frac{1}{h^2}\right]\widehat{w}_{i+1} + \left[\frac{3\mu p_3(\ddot{\eta}_{i-1})}{2h} - \mu p_4(\ddot{\eta}_{i-1}) + \frac{2\lambda p_3(\ddot{\eta}_i)}{2h} - \frac{\mu p_3(\ddot{\eta}_{i+1})}{2h}\right]\hat{y}_{i-1} + \left[\frac{-4\mu p_3(\ddot{\eta}_{i-1})}{2h} - 2\lambda p_4(\ddot{\eta}_i) + \frac{4\mu p_3(\ddot{\eta}_{i+1})}{2h}\right]\hat{y}_i + \left[\frac{\mu p_3(\ddot{\eta}_{i-1})}{2h} - \frac{2\lambda p_3(\ddot{\eta}_i)}{2h} - \frac{3\mu p_3(\ddot{\eta}_{i+1})}{2h} - \mu p_4(x_{i+1})\right]\hat{y}_{i+1} = -\mu S_2(\ddot{\eta}_{i-1},\hat{y}_{i-1},\widehat{w}_{i-1}) - 2\lambda S_2(\ddot{\eta}_i,\hat{y}_i,\widehat{w}_i) - \mu S_2(\ddot{\eta}_{i+1},\hat{y}_{i+1},\widehat{w}_{i+1}) - \mu r_2(\ddot{\eta}_{i-1}) - 2\lambda r_2(\ddot{\eta}_i) - \mu r_2(\ddot{\eta}_{i+1}) \quad (20)$$

We need fuor more equationaution.the four end condition can be derivated as follows :

$$\hat{y}(0) = \hat{y}(1) = 0, \widehat{w}(0) = \widehat{w}(0) = 0 \quad (21)$$

The method is described in matrix form in the following:

$$CY = D \quad (22)$$

Where

$$\hat{Y} = \hat{y}_0, \hat{y}_1, \dots, \hat{y}_{n,} \widehat{w}_0, \widehat{w}_1, \dots, \widehat{w}_n \quad (23)$$

Ultimately, the nonlinear system is solved using Maple22 to yield the approximate solution.

4. Numerical examples

In this section, to illustrate our methods we have solved two non-linear system of second-order boundary value problems. All computations are done by using Maple22.

Example 1. We consider the following equations

$$\hat{y}^{(2)}(\ddot{\eta}) + \hat{y}^{(1)}(\ddot{\eta}) + \ddot{\eta}\hat{y}(\ddot{\eta}) + w^{(1)}(\ddot{\eta}) + 2\ddot{\eta}\hat{y}(\ddot{\eta}) + \ddot{\eta} + 2(\ddot{\eta}+1)\cos(\ddot{\eta}) - \pi\cos(\pi\ddot{\eta}) -$$

$\pi\cos(\pi\ddot{\eta}) - \ddot{\eta}\sin(\pi\ddot{\eta}) - (4\ddot{\eta} - 2\ddot{\eta}^2 - 2) = 0$

$w^{(2)}(\ddot{\eta}) + w(\ddot{\eta}) + 2w^{(1)}(\ddot{\eta}) + x^2\hat{y}(\ddot{\eta}) - 4x\cos(\ddot{\eta}) - 2(-\ddot{\eta}^3 + \ddot{\eta}^2 + \ddot{\eta} + 3)\sin(\ddot{\eta})$
$-(1-\pi^2)\sin(\pi\ddot{\eta}) = 0$

Subject to the boundary conditions:

$\hat{y}(0) = \hat{y}(1) = 0, w(0) = w(0) = 0$, Where $0 < \ddot{\eta} < 1$

The precise solutions are $\hat{y}(\ddot{\eta}) = 2(1-\ddot{\eta})\sin\ddot{\eta}$ and $\widehat{w}(\ddot{\eta}) = \sin\pi\ddot{\eta}$ .Equations (22) yield a system of linear equations for this situation. Table 1 given a comparison of the findings of the non-polynomial exponential spline function, with the exact solution. The absolute errors of our technique for $n = 10$ and along with the findings from [5], [6] using the same number of points in the interval $[0,1]$ and the exact solutions, are reported in Table 2.

Example 2: Consider the linear system of second-order boundary value problems

$\hat{y}^{(2)}(\ddot{\eta}) + \ddot{\eta}\hat{y}(\ddot{\eta}) + 2\ddot{\eta}\widehat{w}(\ddot{\eta}) + \ddot{\eta}\hat{y}^2(\ddot{\eta}) - 2\ddot{\eta}\sin(\pi\ddot{\eta}) - (\ddot{\eta}^5 + 2\ddot{\eta}^4 + \ddot{\eta}^2 - 2) = 0$

$\hat{y}^{(1)}(\ddot{\eta}) + \widehat{w}(\ddot{\eta}) + \ddot{\eta}^2\hat{y}(\ddot{\eta}) + \widehat{w}^2(\ddot{\eta})\sin(\ddot{\eta}) - \ddot{\eta}^3(1-\ddot{\eta}) - \sin(\pi\ddot{\eta})\,(1 + \sin(\ddot{\eta})\sin(\pi\ddot{\eta}))$
$+ \pi\cos(\pi\ddot{\eta}) = 0$

Subject to the boundary conditions

$\hat{y}(0) = \hat{y}(1) = 0, \widehat{w}(0) = \widehat{w}(0) = 0.$

Where $0 < \ddot{\eta} < 1$

The exact solutions are $\hat{y}(\ddot{\eta}) = \ddot{\eta} - \ddot{\eta}^2$ and $\widehat{w}(\ddot{\eta}) = \sin\pi\ddot{\eta}$. In Table 3 presents a comparison between the results obtained using the non-polynomial exponential spline function and the exact solution. Table 4 present a comparison between the errors of our technique, using $n = 10$, and the method described in reference [5], [6]. Both methods use the same number of points inside the interval $[0,1]$ and are compared against the exact answers. It is important to mention that, as far as we know, there is just one publishe d study addressing the solution to the problem mentioned in this particular paper. The authors of [5], [6] have successfully demonstrated the existence and uniqueness of the solution for the model provided in this study. In addition, they incorporated a numerical technique into their research. Therefore, we were only able to make a comparison between the findings produced from our process and those acquired from the technique described in reference [5], [6].

Table 1- Comparison between the exact solutions and proposed method for Example 1

	Exact solution		**Proposed method**		**Absolute errors**	
ξ	$\hat{y}(x)$	$\widehat{w}(\ddot{\eta})$	$\hat{y}(x)$	$\widehat{w}(\ddot{\eta}))$	$\hat{y}(x)$	$\widehat{w}(\ddot{\eta})$
0.0	0.0000000000	0.000000000	0.0000000000	0.0000000000	0.00000000	0.00000000
0.1	0.1797001500	0.3088655201	0.1797001541	0.3088655211	4.1×10^{-9}	1.0×10^{-9}
0.2	0.3178709292	0.5875275257	0.3178709273	0.5875275249	1.9×10^{-9}	8.0×10^{-10}
0.3	0.4137282894	0.8087360606	0.4137282863	0.8087360655	3.1×10^{-9}	4.9×10^{-9}
0.4	0.4673020108	0.9508594605	0.4673020145	0.9508594614	3.7×10^{-9}	9.0×10^{-10}
0.5	0.4794255386	0.9999996829	0.4794255390	0.9999996830	$4\ 1\times 10^{-10}$	1.0×10^{-10}
0.6	0.4517139788	0.9513513762	0.4517139771	0.9513513751	1.7×10^{-9}	1.1×10^{-9}
0.7	0.3865306124	0.8096717883	0.3865306136	0.8096717873	1.2×10^{-9}	1.0×10^{-9}
0.8	0.2869424364	0.5888155620	0.2869424359	0.5888155651	5.1×10^{-9}	3.1×10^{-9}
0.9	0.1566653819	0.3103799097	0.1566653814	0.3103799098	5.1×10^{-10}	1.0×10^{-10}
1.0	0.0000000000	0.000000000	0.0000000000	0.000000000	0.00000000	0.00000000

Table 2- The absolute errors of $\hat{y}(\ddot{\eta})$ and $\widehat{w}(\ddot{\eta})$ for Example 1 with $n = 10$.

	Geng and Cui [5]		**Dehghan et al. [6]**		**Present method**	
ξ	$\hat{y}(x)$	$\widehat{w}(\ddot{\eta})$	$\hat{y}(x)$	$\widehat{w}(\ddot{\eta})$	$\hat{y}(x)$	$\widehat{w}(\ddot{\eta})$
0.08	3.3×10^{-3}	7.7×10^{-3}	3.2×10^{-3}	2.0×10^{-3}	5.2×10^{-10}	2.6×10^{-9}
0.24	7.7×10^{-3}	2.0×10^{-2}	9.2×10^{-4}	9.8×10^{-3}	8.3×10^{-10}	7.5×10^{-9}
0.40	9.7×10^{-3}	2.7×10^{-2}	2.0×10^{-3}	1.1×10^{-3}	4.9×10^{-9}	9.1×10^{-9}
0.56	9.5×10^{-3}	2.7×10^{-2}	2.2×10^{-4}	1.4×10^{-2}	6.5×10^{-10}	3.7×10^{-10}
0.72	7.3×10^{-3}	2.0×10^{-2}	4.1×10^{-3}	5.5×10^{-3}	3.8×10^{-9}	8.2×10^{-10}
0.88	3.4×10^{-3}	9.4×10^{-2}	1.0×10^{-2}	7.7×10^{-2}	9.1×10^{-10}	5.4×10^{-10}
0.96	1.1×10^{-3}	3.1×10^{-3}	2.1×10^{-3}	8.3×10^{-3}	2.7×10^{-10}	1.9×10^{-10}

Table 3- Comparison between the exact solutions and proposed method for Example 2

	Exact solution		**Proposed method**		**Absolute errors**	
ξ	$\hat{y}(x)$	$\widehat{w}(\ddot{\eta})$	$\hat{y}(x)$	$w(x)$	$\hat{y}(x)$	$\widehat{w}(\ddot{\eta})$
0.0	0.00000000	0.000000000	0.000000000	0.000000000	0.000000000	0.000000000
0.1	0.090000000	0.3088655201	0.0900000012	0.3088655211	0.000000000	1.0×10^{-9}
0.2	0.160000000	0.5875275257	0.1600000043	0.5875275249	4.3×10^{-9}	8.0×10^{-10}
0.3	0.210000000	0.8087360606	0.2100000074	0.8087360655	7.4×10^{-9}	4.9×10^{-9}
0.4	0.240000000	0.9508594605	0.2400000033	0.9508594614	3.3×10^{-9}	9.0×10^{-10}
0.5	0.250000000	0.9999996829	0.2500000008	0.9999996830	$8\ 0\times 10^{-9}$	1.0×10^{-10}
0.6	0.240000000	0.9513513762	0.2400000062	0.9513513751	6.2×10^{-9}	1.1×10^{-9}
0.7	0.210000000	0.8096717883	0.2100000051	0.8096717873	5.1×10^{-9}	1.0×10^{-9}
0.8	0.160000000	0.5888155620	0.1600000045	0.5888155651	4.5×10^{-9}	3.1×10^{-9}
0.9	0.090000000	0.3103799097	0.0900000000	0.3103799098	0.0000000	1.0×10^{-10}
1.0	0.00000000	0.000000000	0.000000000	0.000000000	0.0000000	0.00000000

Table 4- The absolute errors of $\hat{y}(\ddot{\eta})$ and $\widehat{w}(\ddot{\eta})$for Example 2 with $n = 10$.

	Geng and Cui [5]		**Dehghan et al. [6]**		**Present method**	
ξ	$\hat{y}(x)$	$\widehat{w}(\ddot{\eta})$	$\hat{y}(x)$	$\widehat{w}(\ddot{\eta})$	$\hat{y}(x)$	$\widehat{w}(\ddot{\eta})$
0.08	8.0×10^{-3}	1.9×10^{-3}	3.0×10^{-4}	2.0×10^{-3}	6.3×10^{-10}	7.4×10^{-10}
0.24	1.9×10^{-3}	5.1×10^{-3}	8.5×10^{-3}	9.8×10^{-4}	4.5×10^{-10}	5.2×10^{-10}
0.40	2.4×10^{-3}	7.1×10^{-3}	3.5×10^{-3}	1.1×10^{-3}	7.6×10^{-10}	7.5×10^{-9}
0.56	2.4×10^{-3}	6.9×10^{-3}	2.6×10^{-3}	1.4×10^{-3}	3.9×10^{-10}	2.9×10^{-10}
0.72	1.8×10^{-3}	5.2×10^{-3}	2.0×10^{-3}	5.5×10^{-5}	5.2×10^{-10}	6.3×10^{-10}
0.88	8.0×10^{-3}	2.4×10^{-3}	2.6×10^{-3}	7.7×10^{-4}	8.4×10^{-10}	1.6×10^{-10}
0.96	2.0×10^{-3}	8.0×10^{-4}	2.6×10^{-3}	8.3×10^{-4}	5.1×10^{-10}	5.9×10^{-10}

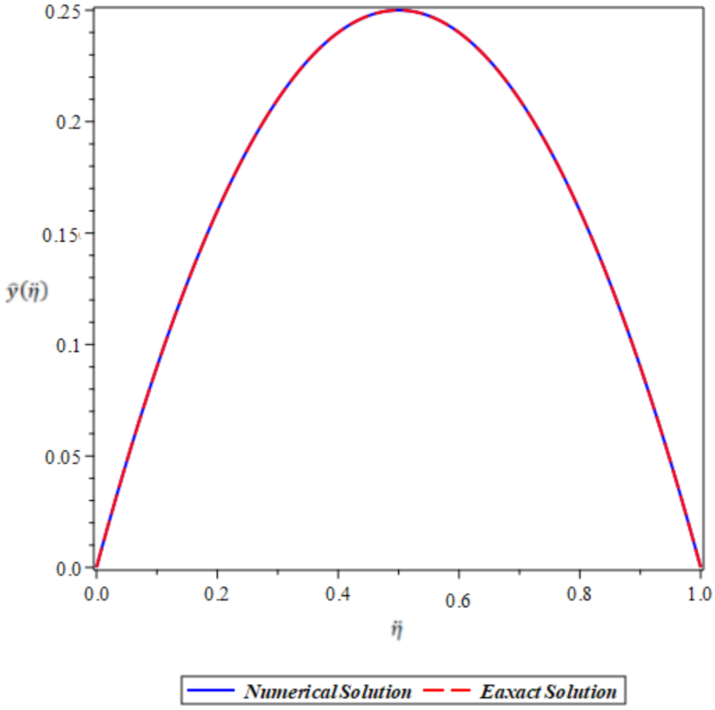

Figure -1 Comparing the numerical solution and exact solution of $\hat{y}(\ddot{\eta})$ in Example 1

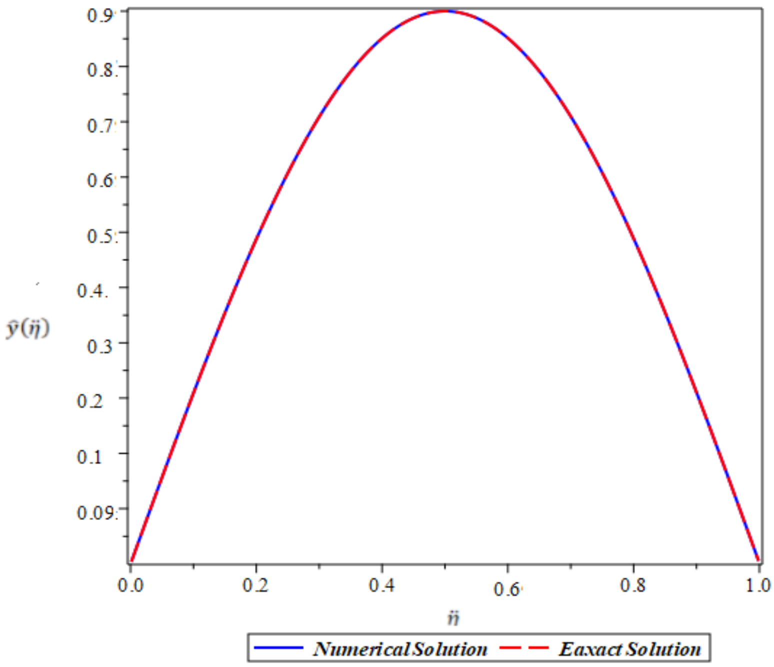

Figure -2 Comparing the numerical solution and exact solution of $\widehat{w}(\ddot{\eta})$ in Example 1

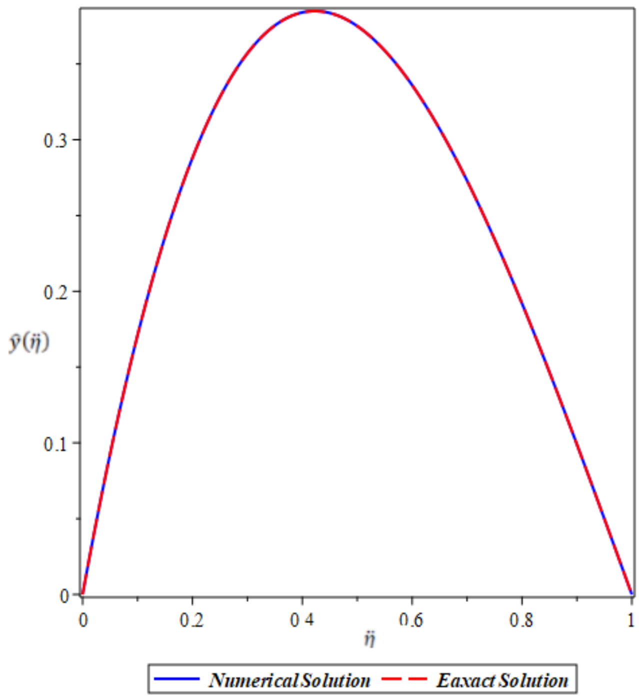

Figure -3 Comparing the numerical solution and exact solution of $\hat{y}(\ddot{\eta})$ in Example 2

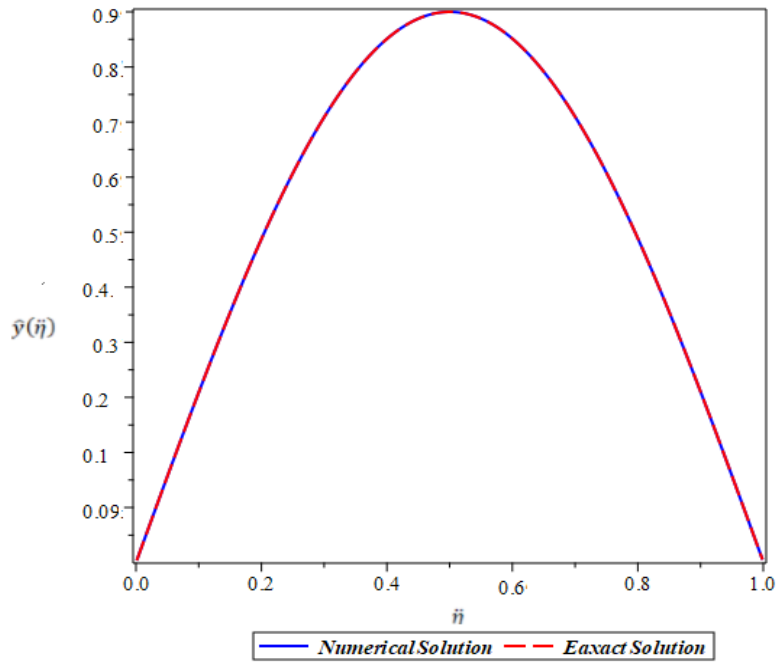

Figure -4 Comparing the numerical solution and exact solution of $\widehat{w}(\ddot{\eta})$ in Example 2

5. Conclusions

This work presents the development of the non-polynomial exponential spline approach for approximating the solution of second-order boundary value problems in nonlinear systems. The numerical outcomes acquired through the utilization of the method outlined in this investigation yield satisfactory results. Our analysis has determined that the numerical findings approach the exact solution when the value of h approaches zero. The findings depicted in Figures 1, 2, 3, and 4 demonstrate that as the value of n rose, the greatest absolute error dropped. The utilization of the spline method has demonstrated its efficacy as a viable approach for solving systems of boundary value problems.

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