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On Classes of Analytic Multivalent Functions Involving by Generalization of Differential Operator with Negative Coefficients

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Abstract

Let Ą(ꝑ) be the class of analytic and *p*-valent functions with negative coefficients .Also let $\Gamma^*(ꝑ,\lambda,\phi,\delta,\alpha)$ and $Ꝁ\Gamma^*(ꝑ,\lambda,\phi,\delta,\alpha)$ denote the subclasses of Ą(ꝑ) consisting of *p*-valent analytic functions with negative coefficients in the open unit disk . The object of the present paper is to prove distortion theorems of functions in the classes $\Gamma^*(ꝑ,\lambda,\phi,\delta,\alpha)$ and $Ꝁ\Gamma^*(ꝑ,\lambda,\phi,\delta,\alpha)$,coefficient estimates , extreme points, closure theorems, and inclusion properties. Also, we obtain radii of convexity, starlikeness and close-to-convexity for functions belonging to those classes. By specializing the parameters involved, the corresponding results for several interesting subclasses of analytic functions can easily be deduced.

Keywords: Multivalent functions Differential operator, Fractional calculus and Negative Coefficients.

حول اصناف الدوال التحليلية المتعددة التكافؤ المتضمنة المؤثرات التفاضلية المعممة ذات المعاملات السالبة

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الخلاصة

نفرض Ą(ꝑ) هو صنف من الدوال التحليلية متعددة التكافؤ ذات المعاملات السالبة . كذلك نفرض $\Gamma^*(ꝑ,\lambda,\phi,\delta,\alpha)$ و $Ꝁ\Gamma^*(ꝑ,\lambda,\phi,\delta,\alpha)$ تشير إلى الاصناف الفرعية لـ Ą(ꝑ) التي تتكون من دوال تحليلية متعددة التكافؤ مع المعاملات السالبة في قرص الوحدة المفتوحة . الهدف من هذا البحث هو إثبات نظريات التشويه للدوال للاصناف $\Gamma^*(ꝑ,\lambda,\phi,\delta,\alpha)$ و $Ꝁ\Gamma^*(ꝑ,\lambda,\phi,\delta,\alpha)$, معامل التقديرات، النقاط القصوى، نظريات الإغلاق، وخصائص التضمين. كما حصلنا على نصف قطر التحدب والشبه النجمي والقريب من التحدب للدوال التي تنتمي إلى تلك الاصناف. من خلال تخصيص المعاملات المعنية، يمكن بسهولة استخلاص النتائج المتعلقة بالاصناف الفرعية المثيرة للاهتمام من الدوال التحليلية.

1. Introduction

Let $\mathrm{A}(p)$ denoted the class which contains the functions $h(z)$, where

$$h(z) = z^{p} - \sum_{n=p+j}^{\infty} a_n z^n \quad , \quad a_n \geq 0\,, p, j \in N = \{1,2,\ldots\}\,, \tag{1}$$

which are regular and p-valent in the open unit disk $U = \{z: |z| < 1\}$. A function $h(z) \in \mathrm{A}(p)$ is said to be p-valently starlike of order α if it satisfies the inequality

$$Re\left\{\frac{zh'(z)}{h(z)}\right\} > \alpha\,, (\, z \in U; 0 \leq \alpha < p; p \in N).$$

We denote by $S^*(p,\alpha)$ the class of all p-valently starlike functions of order α

Also , a function $h(z) \in \mathrm{A}(p)$ is said to be p-valently convex of order α if it satisfies the inequality

$$Re\left\{1 + \frac{zh''(z)}{h'(z)}\right\} > \alpha\,, (\, z \in U; 0 \leq \alpha < p; p \in N).$$

We denote by $C(p,\alpha)$ the class of all p-valently convex functions of order α.

The classes $S^*(p,\alpha)$ and $C(p,\alpha)$ are studied by by Owa [9] .

For each $h(z) \in \mathrm{A}(p)$, we have

$$h^q(z) = \frac{p!}{(p-q)!} z^{p-q} - \sum_{n=p+j}^{\infty} \frac{n!}{(n-q)!} a_n z^{n-q}, (q \in N_0 = N \cup \{0\};\ p > q)$$

In [1] , Aouf Define the following for each $h(z) \in \mathrm{A}(p)$,

$$D_p^r h^q(z) = \frac{p!}{(p-q)!} z^{p-q}$$
$$- \sum_{n=p+j}^{\infty} \frac{n!}{(n-q)!} \left(\frac{n-q}{p-q}\right)^r a_n z^{n-q}, (p, j \in N; q \in N_0 = N \cup \{0\}; p > q).$$

We note that , when q=0 and p=1 , the differential operator $D_1^r = D^r$ is introduced by Salagean [11].

Now , we introduce the following definition

Definition 1 : A function $h(z) \in \Gamma(p, \lambda, \phi, \delta, \alpha)$ if and only if

$$\left| \frac{\delta z\left(D_p^r h^{q+1}(z)\right) + \lambda z^2\left(D_p^r h^{q+2}(z)\right)}{(1-\lambda)\left(D_p^r h^q(z)\right) + z\left(D_p^r h^{q+1}(z)\right)} - (\delta - \phi) \right| < \alpha$$

$z \in E, q \in \mathbb{N}U\{0\}, 0 < \alpha \leq 1, \phi \in \mathbb{R}, \phi < 1, p > q, \gamma, \delta \leq 1\,, 0 \leq \lambda \leq 1.$

In addition a function $h(z) \in \text{Ꝁ}\Gamma(p, \lambda, \phi, \delta, \alpha)$ if and $zh'(z) \in \Gamma(p, \lambda, \phi, \delta, \alpha)$.

T(ꝑ) is the subclass of Ą(ꝑ) that contains functions of the form

$$h(z) = z^{ꝑ} - \sum_{n=ꝑ+j}^{\infty} a_n z^n, a_n \geq 0. \qquad (2)$$

The classes $\Gamma^*(ꝑ,\lambda,\phi,\delta,\alpha)$ and $ꝀΓ^*(ꝑ,\lambda,\phi,\delta,\alpha)$ intersect with T(p).

$Đ_z^q h(z)$ is the q^{th} order differential operator for h(w) ∈ Ą(ᵽ) defined in (1)

$$D_p^r h^q(z) = \frac{ꝑ!}{(ꝑ-q)!} z^{ꝑ-q} - \sum_{n=ꝑ+j}^{\infty} \frac{n!}{(n-q)!}\left(\frac{n-q}{ꝑ-q}\right)^r a_n z^{n-q}, \quad ꝑ > q.$$

Many authors were studied analytic classes such as [2,3,4,5,6,7,8,10] and get many results for various classes.

Theorem 1 : Let $h(z)$ be in $\Gamma^*(ꝑ,\lambda,\phi,\delta,\alpha)$ if and only if

$$\sum_{n=ꝑ+1}^{\infty} \frac{1}{Ɛ(n)} a_n \leq 1, \qquad (3)$$

where

$$Ɛ(n) = \frac{\frac{ꝑ!}{(ꝑ-q)!}[\alpha(1-\lambda+ꝑ-q) - \lambda(ꝑ-q)(ꝑ-q-1) - \phi(ꝑ-q) + (\delta-\phi)(1-\lambda)]}{\frac{n!}{(n-q)!}\left(\frac{n-q}{ꝑ-q}\right)^r [\lambda(n-q)(n-q-1) + \phi(n-q) - (\delta-\phi)(1-\lambda) + \alpha(1-\lambda+n-q)]}$$

Proof: Let $h(z) \in$ in $\Gamma^*(ꝑ,\lambda,\phi,\delta,\alpha)$. Thus

$$\left| \frac{\delta z\left(D_p^r h^{q+1}(z)\right) + \lambda z^2\left(D_p^r h^{q+2}(z)\right)}{(1-\lambda)\left(D_p^r h^q(z)\right) + z\left(D_p^r h^{q+1}(z)\right)} - (\delta-\phi) \right| < \alpha. \qquad (4)$$

Consider

$$\delta z\left(D_p^r h^{q+1}(z)\right) + \lambda z^2\left(D_p^r h^{q+2}(z)\right) - (\delta - \phi)\left[(1-\lambda)\left(D_p^r h^q(z)\right) + z\left(D_p^r h^{q+1}(z)\right)\right]$$

$$= \frac{ꝑ!}{(ꝑ-q)!}[\lambda(ꝑ-q)(ꝑ-q-1) + \phi(ꝑ-q) - (\delta-\phi)(1-\lambda)]z^{ꝑ-q} - \sum_{n=ꝑ+j}^{\infty} \frac{n!}{(n-q)!}\left(\frac{n-q}{ꝑ-q}\right)^r [\lambda(n-q)(n-q-1) + \phi(n-q) - (\delta-\phi)(1-\lambda)]a_n z^{n-q}$$

Consider now

$$(1-\lambda)\left(D_p^r h^q(z)\right) + z\left(D_p^r h^{q+1}(z)\right)$$

$$= \frac{ꝑ!}{(ꝑ-q)!}[1-\lambda+ꝑ-q]z^{ꝑ-q} - \sum_{n=ꝑ+1}^{\infty} \frac{n!}{(n-q)!}\left(\frac{n-q}{ꝑ-q}\right)^r [1-\lambda+n-q]a_n z^{n-q}$$

From (3)

$$\left|\frac{\frac{\mathrm{p}!}{(\mathrm{p}-q)!}[\lambda(\mathrm{p}-q)(\mathrm{p}-q-1)+\phi(\mathrm{p}-q)-(\delta-\phi)(1-\lambda)]z^{\mathrm{p}-q}-\sum_{n=\mathrm{p}+1}^{\infty}\frac{n!}{(n-q)!}\left(\frac{n-q}{\mathrm{p}-q}\right)^{r}[\lambda(n-q)(n-q-1)+\phi(n-q)-(\delta-\phi)(1-\lambda)]a_n z^{n-q}}{\frac{\mathrm{p}!}{(\mathrm{p}-q)!}[1-\lambda+\mathrm{p}-q]z^{\mathrm{p}-q}-\sum_{n=\mathrm{p}+1}^{\infty}\frac{n!}{(n-q)!}\left(\frac{n-q}{\mathrm{p}-q}\right)^{r}[1-\lambda+n-q]a_n z^{n-q}}\right|\leq\propto. \quad (5)$$

Since $|\mathrm{Re}\,(z)|\leq|z|$, we can choose values on the real axis. In (5) Allowing $z\to 1^{-}$ through real axis, it follows that

$$\frac{\mathrm{p}!}{(\mathrm{p}-q)!}[\lambda(\mathrm{p}-q)(\mathrm{p}-q-1)+\phi(\mathrm{p}-q)-(\delta-\phi)(1-\lambda)]+$$

$$\sum_{n=\mathrm{p}+j}^{\infty}\frac{n!}{(n-q)!}\left(\frac{n-q}{\mathrm{p}-q}\right)^{r}[\lambda(n-q)(n-q-1)+\phi(n-q)-(\delta-\phi)(1-\lambda)]a_n$$

$$\leq\alpha\left[\frac{\mathrm{p}!}{(\mathrm{p}-q)!}[1-\lambda+\mathrm{p}-q]z^{\mathrm{p}-q}-\sum_{n=\mathrm{p}+j}^{\infty}\frac{n!}{(n-q)!}\left(\frac{n-q}{\mathrm{p}-q}\right)^{r}[1-\lambda+n-q]a_n\right].$$

Therefore

$$\sum_{n=\mathrm{p}+j}^{\infty}\frac{n!}{(n-q)!}\left(\frac{n-q}{\mathrm{p}-q}\right)^{r}[\lambda(n-q)(n-q-1)+\phi(n-q)-(\delta-\phi)(1-\lambda)$$
$$+\alpha(1-\lambda+n-q)]a_n$$
$$\leq\frac{\mathrm{p}!}{(\mathrm{p}-q)!}[\alpha(1-\lambda+\mathrm{p}-q)-\lambda(\mathrm{p}-q)(\mathrm{p}-q-1)-\phi(\mathrm{p}-q)$$
$$+(\delta-\phi)(1-\lambda)].$$

Define

$$\mathcal{E}(n)=\frac{\frac{\mathrm{p}!}{(\mathrm{p}-q)!}\begin{bmatrix}\alpha(1-\lambda+\mathrm{p}-q)-\lambda(\mathrm{p}-q)(\mathrm{p}-q-1)-\phi(\mathrm{p}-q)\\+(\delta-\phi)(1-\lambda)\end{bmatrix}}{\frac{n!}{(n-q)!\left(\frac{n-q}{\mathrm{p}-q}\right)^{r}\begin{bmatrix}\lambda(n-q)(n-q-1)+\phi(n-q)-(\delta-\phi)(1-\lambda)\\+\alpha(1-\lambda+n-q)\end{bmatrix}}}$$

We get

$$\sum_{n=\mathrm{p}+j}^{\infty}\frac{1}{\mathcal{E}(n)}a_n\leq 1.$$

Now , Suppose that (3) is true. We have

$$\left|\delta z\left(D_p^r h^{q+1}(z)\right)+\lambda z^2\left(D_p^r h^{q+2}(z)\right)-(\delta-\phi)(1-\lambda)\left(D_p^r h^{q}(z)\right)\right.$$
$$\left.+z\left(Đ_z^{q+1}\left(\Omega_{\mathrm{p}}(r,\mathrm{p})h(z)\right)\right)\right|-\propto\left|(1-\lambda)\left(D_p^r h^{q}(z)\right)+z\left(D_p^r h^{q+1}(z)\right)\right|$$

$$= \sum_{n=\mathrm{p}+j}^{\infty} \frac{n!}{(n-q)!}\left(\frac{n-q}{\mathrm{p}-q}\right)^{r} [\lambda(n-q)(n-q-1)+\phi(n-q)-(\delta-\phi)(1-\lambda)+\alpha(1-\lambda+n-q)]a_n - \frac{\mathrm{p}!}{(\mathrm{p}-q)!}[\alpha(1-\lambda+\mathrm{p}-q)-\lambda(\mathrm{p}-q)(\mathrm{p}-q-1)-\phi(\mathrm{p}-q)+(\delta-\phi)(1-\lambda)] \leq 0.$$

By using maximum modulus principle and (3) , $h(z) \in \Gamma^*(\mathrm{p},\lambda,\phi,\delta,\alpha)$.

Corollary 1: If $h(z) \in \Gamma^*(\mathrm{p},\lambda,\phi,\delta,\alpha)$, then

$$a_n \leq \Sigma(n)$$

For $n = \mathrm{p}+1, \mathrm{p}+2, \ldots$.

Theorem 2 :Suppose that $h(z) \in Ʞ\Gamma^*(\mathrm{p},\lambda,\phi,\delta,\alpha)$ if and only if $\sum_{n=\mathrm{p}+1}^{\infty} \frac{n}{\Sigma(n)} a_n \leq \mathrm{p}$

Proof :Let $h(z) \in Ʞ\Gamma^*(\mathrm{p},\lambda,\phi,\delta,\alpha)$ therefore $zf'(z) \in Ʞ\Gamma^*(\mathrm{p},\lambda,\phi,\delta,\alpha)$

Substitute $h(z) = zh'(z)$

$$h(z) = \mathrm{p}z^{\mathrm{p}} - \sum_{n=\mathrm{p}+j}^{\infty} n a_n z^n$$

Thus

$$\left| \frac{\delta z\left(D_p^r h^{q+1}(z)\right) + \lambda z^2\left(D_p^r h^{q+2}(z)\right)}{(1-\lambda)\left(D_p^r h^q(z)\right) + z\left(D_p^r h^{q+1}(z)\right)} - (\delta-\phi) \right| < \alpha \qquad (6)$$

Consider

$$\delta z\left(D_p^r h^{q+1}(z)\right) + \lambda z^2\left(D_p^r h^{q+2}(z)\right) - (\delta - \phi)\left[(1-\lambda)\left(D_p^r h^q(z)\right) + z\left(D_p^r h^{q+1}(z)\right)\right]$$

$$= \frac{\mathrm{p}!\,\mathrm{p}}{(\mathrm{p}-q)!}[\lambda(\mathrm{p}-q)(\mathrm{p}-q-1)+\phi(\mathrm{p}-q)-(\delta-\phi)(1-\lambda)]z^{\mathrm{p}-q} - \sum_{n=\mathrm{p}+j}^{\infty} \frac{n!\,n}{(k-q)!}\left(\frac{n-q}{\mathrm{p}-q}\right)^{r}[\lambda(n-q)(n-q-1)+\phi(n-q)-(\delta-\phi)(1-\lambda)]a_n z^{n-q}$$

Now, consider

$$(1-\lambda)\left(D_p^r h^q(z)\right) + z\left(D_p^r h^{q+1}(z)\right)$$

$$=\frac{\mathrm{p}!\mathrm{p}}{(\mathrm{p}-q)!}[1-\lambda+\mathrm{p}-q]z^{\mathrm{p}-q} - \sum_{n=\mathrm{p}+j}^{\infty} \frac{n!\,n}{(k-q)!}\left(\frac{n-q}{\mathrm{p}-q}\right)^{r}[1-\lambda+n-q]a_n z^{n-q}.$$

From (6), we have

$$\left|\frac{\frac{\mathfrak{p}!\,\mathfrak{p}}{(\mathfrak{p}-q)!}[\lambda(\mathfrak{p}-q)(\mathfrak{p}-q-1)+\phi(\mathfrak{p}-q)-(\delta-\phi)(1-\lambda)]z^{\mathfrak{p}-q}-\sum_{n=\mathfrak{p}+j}^{\infty}\frac{n!\,n}{(k-q)!}\left(\frac{n-q}{\mathfrak{p}-q}\right)^{r}[\lambda(n-q)(n-q-1)+\phi(n-q)-(\delta-\phi)(1-\lambda)]a_n z^{n-q}}{\frac{\mathfrak{p}!\,\mathfrak{p}}{(\mathfrak{p}-q)!}[1-\lambda+\mathfrak{p}-q]z^{\mathfrak{p}-q}-\sum_{n=\mathfrak{p}+j}^{\infty}\frac{n!\,n}{(n-q)!}\left(\frac{n-q}{\mathfrak{p}-q}\right)^{r}[1-\lambda+n-q]a_n z^{n-q}}\right|$$
$$\leq \propto. \qquad (7)$$

Given that $|\mathrm{Re}\,(z)| \leq |z|$ and that $z\to 1^-$ passes through the real axis, we obtain

$$\sum_{n=\mathfrak{p}+j}^{\infty}\frac{n!\,n}{(n-q)!}\left(\frac{n-q}{\mathfrak{p}-q}\right)^{r}[\lambda(n-q)(n-q-1)+\phi(n-q)-(\delta-\phi)(1-\lambda)+\alpha(1-\lambda+n-q)]a_n$$
$$\leq \frac{\mathfrak{p}!\,\mathfrak{p}}{(\mathfrak{p}-q)!}[\alpha(1-\lambda+\mathfrak{p}-q)-\lambda(\mathfrak{p}-q)(\mathfrak{p}-q-1)-\phi(\mathfrak{p}-q)+(\delta-\phi)(1-\lambda)]$$

Therefore ,we get $\sum_{n=\mathfrak{p}+j}^{\infty}\frac{n}{\Sigma(n)}a_n \leq \mathfrak{p}$.

Corollary 2: If $h(z) \in Ꝃ\Gamma^*(\mathfrak{p},\lambda,\phi,\delta,\alpha)$, then $a_n \leq \frac{\mathfrak{p}}{n}\Sigma(n)$, for $n = \mathfrak{p}+1, \mathfrak{p}+2, \ldots$.

Theorem 3: If $h(z) \in \Gamma^*(\mathfrak{p},\lambda,\phi,\delta,\alpha)$, then

$$|z|^{\mathfrak{p}} - \Sigma(\mathfrak{p}+1)|z|^{1+\mathfrak{p}} \leq |h(z)| \leq |z|^{\mathfrak{p}} + \Sigma(\mathfrak{p}+1)|z|^{1+\mathfrak{p}}.$$

Proof : $h(z) \in \Gamma^*(\mathfrak{p},\lambda,\phi,\delta,\alpha)$ if and only if $\sum_{n=\mathfrak{p}+j}^{\infty}\frac{1}{\Sigma(n)}a_n \leq 1$

$$|h(z)| \leq |z^{\mathfrak{p}}| + \left|\sum_{n=\mathfrak{p}+j}^{\infty} a_n z^n\right| \leq |z|^{\mathfrak{p}} + \Sigma(\mathfrak{p}+1)|z|^{1+\mathfrak{p}}. \qquad (8)$$

Similarly

$$|h(z)| \geq |z^{\mathfrak{p}}| - \left|\sum_{n=\mathfrak{p}+j}^{\infty} a_n z^n\right| \geq |z|^{\mathfrak{p}} - \Sigma(\mathfrak{p}+1)|z|^{1+\mathfrak{p}}. \qquad (9)$$

From (8) and (9), we have

$$|z|^{\mathfrak{p}} - \Sigma(\mathfrak{p}+1)|z|^{1+\mathfrak{p}} \leq |h(z)| \leq |z|^{\mathfrak{p}} + \Sigma(\mathfrak{p}+1)|z|^{1+\mathfrak{p}}.$$

Theorem 4: If $h(z) \in Ꝃ\Gamma^*(\mathfrak{p},\lambda,\phi,\delta,\alpha)$, then

$$\left|z|^{\mathfrak{p}} - \left(\frac{\mathfrak{p}}{\mathfrak{p}+1}\right)\Sigma(\mathfrak{p}+1)\right| z|^{1+\mathfrak{p}} \leq |h(z)| \leq \left|z|^{\mathfrak{p}} + \left(\frac{\mathfrak{p}}{\mathfrak{p}+1}\right)\Sigma(\mathfrak{p}+1)\right| z|^{1+\mathfrak{p}}.$$

Proof : Suppose that $h(z) \in \Gamma^*(\mathfrak{p},\lambda,\phi,\delta,\alpha)$ if and only if $\sum_{n=\mathfrak{p}+1}^{\infty}\frac{n}{\Sigma(n)}a_n \leq \mathfrak{p}$.

$$|h(z)| \leq |z^{\mathfrak{p}}| + \left|\sum_{n=\mathfrak{p}+j}^{\infty} a_n z^n\right|$$
$$\leq \left|z|^{\mathfrak{p}} + \left(\frac{\mathfrak{p}}{\mathfrak{p}+1}\right)\Sigma(\mathfrak{p}+1)\right| z|^{1+\mathfrak{p}}. \qquad (10)$$

Similarly

$$|h(z)| \geq |z^{\mathfrak{p}}| - \left|\sum_{n=\mathfrak{p}+j}^{\infty} a_n z^n\right| \geq \left|z|^{\mathfrak{p}} - \left(\frac{\mathfrak{p}}{\mathfrak{p}+1}\right) \mathcal{E}(\mathfrak{p}+1)\right| z|^{1+\mathfrak{p}}. \qquad (11)$$

From (10) and (11) we have

$$\left|z|^{\mathfrak{p}} - \left(\frac{\mathfrak{p}}{\mathfrak{p}+1}\right) \mathcal{E}(\mathfrak{p}+1)\right| z|^{1+\mathfrak{p}} \leq |h(z)| \leq \left|z|^{\mathfrak{p}} + \left(\frac{\mathfrak{p}}{\mathfrak{p}+1}\right) \mathcal{E}(\mathfrak{p}+1)\right| z|^{1+\mathfrak{p}}.$$

Theorem 5 : If $h(z) \in \Gamma^*(\mathfrak{p}, \lambda, \phi, \delta, \alpha)$, then

$$\mathfrak{p}|z|^{\mathfrak{p}-1} + (\mathfrak{p}+1)\, \mathcal{E}(\mathfrak{p}+1)|z|^{\mathfrak{p}} \leq |h'(z)| \leq \mathfrak{p}|z|^{\mathfrak{p}-1} + (\mathfrak{p}+1)\, \mathcal{E}(\mathfrak{p}+1)|z|^{\mathfrak{p}}$$

Proof : Let $h(z) \in \Gamma^*(\mathfrak{p}, \lambda, \phi, \delta, \alpha)$. Therefore $\sum_{n=\mathfrak{p}+j}^{\infty} \frac{1}{\mathcal{E}(n)} a_n \leq 1$.

$$h'(z) = \mathfrak{p} z^{\mathfrak{p}-1} - \sum_{n=\mathfrak{p}+1}^{\infty} n a_n z^{n-1}$$

$$|h'(z)| \leq |\mathfrak{p} z^{\mathfrak{p}-1}| + \left|\sum_{n=\mathfrak{p}+j}^{\infty} n a_n z^{n-1}\right|$$

$$|h'(z)| \leq \mathfrak{p}|z|^{\mathfrak{p}-1} + (\mathfrak{p}+1)\mathcal{E}(\mathfrak{p}+1)|z|^{\mathfrak{p}}. \qquad (12)$$

Similarly

$$|h'(z)| \geq \mathfrak{p}|z|^{\mathfrak{p}-1} - (\mathfrak{p}+1)\mathcal{E}(\mathfrak{p}+1)|z|^{\mathfrak{p}}. \qquad (13)$$

From (12) and (13) , we have

$$\mathfrak{p}|z|^{\mathfrak{p}-1} + (\mathfrak{p}+1)\, \mathcal{E}(\mathfrak{p}+1)|z|^{\mathfrak{p}} \leq |h'(z)| \leq \mathfrak{p}|z|^{\mathfrak{p}-1} + (\mathfrak{p}+1)\, \mathcal{E}(\mathfrak{p}+1)|z|^{\mathfrak{p}}.$$

Theorem 6 : If $h(z) \in \mathcal{K}\Gamma^*(\mathfrak{p}, \lambda, \phi, \delta, \alpha)$, then

$$\mathfrak{p}\left|z|^{\mathfrak{p}-1} - \left(\frac{\mathfrak{p}}{\mathfrak{p}+1}\right) \mathcal{E}(\mathfrak{p}+1)\right| z|^{\mathfrak{p}} \leq |h'(z)| \leq \mathfrak{p}\left|z|^{\mathfrak{p}-1} + \left(\frac{\mathfrak{p}}{\mathfrak{p}+1}\right) \mathcal{E}(\mathfrak{p}+1)\right| z|^{\mathfrak{p}}$$

Proof : $h(z) \in \mathcal{K}\Gamma^*(\mathfrak{p}, \lambda, \phi, \delta, \alpha)$. Therefore $\sum_{n=\mathfrak{p}+j}^{\infty} \frac{n}{\mathcal{E}(n)} a_n \leq \mathfrak{p}$.

$$h'(z) = \mathfrak{p} z^{\mathfrak{p}-1} - \sum_{n=\mathfrak{p}+j}^{\infty} n a_n z^{n-1}$$

$$|h'(z)| \leq |\mathfrak{p} z^{\mathfrak{p}-1}| + \left|\sum_{n=\mathfrak{p}+j}^{\infty} n a_n z^{n-1}\right|$$

$$|h'(z)| \leq \mathfrak{p}|z|^{\mathfrak{p}-1} + \left(\frac{\mathfrak{p}}{\mathfrak{p}+1}\right) \mathcal{E}(\mathfrak{p}+1)|z|^{\mathfrak{p}}. \qquad (13)$$

Similarly

$$|h'(z)| \geq \mathfrak{p}\left|z|^{\mathfrak{p}-1} - \left(\frac{\mathfrak{p}}{\mathfrak{p}+1}\right) \mathcal{E}(\mathfrak{p}+1)\right| z|^{\mathfrak{p}}. \qquad (14)$$

From (13) and (14) , we have

$$\mathrm{p}\left|z|^{\mathrm{p}-1}-\left(\frac{\mathrm{p}}{\mathrm{p}+1}\right)\,\Sigma(\mathrm{p}+1)\right|z|^{\mathrm{p}}\leq|h'(z)|\leq\mathrm{p}\left|z|^{\mathrm{p}-1}+\left(\frac{\mathrm{p}}{\mathrm{p}+1}\right)\,\Sigma(\mathrm{p}+1)\right|z|^{\mathrm{p}}$$

Theorem 7: If $h(z)\in\Gamma^*(\mathrm{p},\lambda,\phi,\delta,\alpha)$ then $h(z)\in\mathcal{K}(\xi)$ in $|z|<R_1$, where

$$R_1=\int_{n\geq 1+\mathrm{p}}^{\inf}\left\{\left\{\frac{(\mathrm{p}-\xi)}{n\,\Sigma(n)}\right\}^{\frac{1}{n+1-\mathrm{p}}}\right\}.$$

Proof: It is sufficient to show that $\left|\frac{h'(z)}{z^{\mathrm{p}-1}}-\mathrm{p}\right|\leq\mathrm{p}-\xi$ for $|z|<R_1$.

We have

$$\left|\frac{h'(z)}{z^{\mathrm{p}-1}}-\mathrm{p}\right|=\left|-\sum_{n=\mathrm{p}+j}^{\infty}na_nz^{n-\mathrm{p}+1}\right|\leq\sum_{n=\mathrm{p}+j}^{\infty}n|a_n||z|^{n-\mathrm{p}+1}.$$

Thus

$$\sum_{n=\mathrm{p}+j}^{\infty}\frac{n}{(\mathrm{p}-\xi)}|a_n||z|^{n-\mathrm{p}+1}\leq 1. \qquad (15)$$

By Theorem 1

$$\sum_{n=\mathrm{p}+j}^{\infty}\frac{1}{\Sigma(n)}|a_n|\leq 1. \qquad (16)$$

Hence (15) will be true if $\frac{n}{(\mathrm{p}-\xi)}|z|^{n-\mathrm{p}+1}\leq\frac{1}{\Sigma(n)}$

We obtain $|z|\leq\left\{\frac{(\mathrm{p}-\xi)}{n\,\Sigma(n)}\right\}^{\frac{1}{n+1-\mathrm{p}}}$.

Theorem 8 : If $h(z)\in\Gamma^*(\mathrm{p},\lambda,\phi,\delta,\alpha)$ then $h(z)\in\boldsymbol{S}^*(\xi)$ in $|z|<R_2$, where

$$R_2=\underset{n\geq 1+\mathrm{p}}{\overset{inf}{}}\left\{\left\{\left(\left(\frac{\mathrm{p}-\xi}{n-\xi}\right)\frac{1}{\Sigma(n)}\right)^{\frac{1}{n-\mathrm{p}}}\right\}\right\}.$$

Proof : We need to show that $\left|\frac{zh'(z)}{h(z)}-\mathrm{p}\right|\leq\mathrm{p}-\xi$

We have

$$\left|\frac{zh'(z)}{h(z)}-\mathrm{p}\right|\leq\frac{\sum_{n=\mathrm{p}+j}^{\infty}(n-\mathrm{p})|a_n||z|^{n-\mathrm{p}}}{1-\sum_{n=\mathrm{p}+j}^{\infty}|a_n||z|^{n-\mathrm{p}}}\leq\mathrm{p}-\xi. \qquad (17)$$

Hence (17) holds true if

$$\sum_{n=\mathrm{p}+j}^{\infty}\frac{(n-\xi)}{(\mathrm{p}-\xi)}|a_n||z|^{n-\mathrm{p}}\leq 1, \qquad (18)$$

By Theorem 1

$$\sum_{n=\mathfrak{p}+j}^{\infty} \frac{1}{\Sigma(n)} |a_n| \leq 1. \qquad (19)$$

by (18) and (19) will be true if

$$\frac{(n-\xi)}{(\mathfrak{p}-\xi)} |z|^{n-\mathfrak{p}} \leq \frac{1}{\Sigma(n)}$$

$$|z|^{n-\mathfrak{p}} \leq \left(\frac{\mathfrak{p}-\xi}{n-\xi}\right) \frac{1}{\Sigma(n)}$$

THEOREM 9 : If $h(z) \in \Gamma^*(\mathfrak{p}, \lambda, \phi, \delta, \alpha)$, then $h(z) \in C(\xi)$ in $|z| < R_3$, where

$$R_3 = \inf_{n \geq 1+\mathfrak{p}} \left\{ \left\{ \left(\left(\frac{\mathfrak{p}(\mathfrak{p}-\xi)}{n(n-\xi)} \right) \frac{1}{\Sigma(n)} \right\}^{\frac{1}{n-\mathfrak{p}}} \right\} \right\}$$

Proof : We know that $h(z)$ is convex if and only if $h'(w)$ is starlike. We must show that

$$\left| \frac{zh'(z)}{h(z)} - \mathfrak{p} \right| \leq \mathfrak{p} - \xi,$$

where $h(z) = zh'(z)$

$$\left| \frac{zh'(z)}{h(z)} - \mathfrak{p} \right| = \left| \frac{-\sum_{n=\mathfrak{p}+1}^{\infty} n(n-\mathfrak{p}) a_n z^n}{\mathfrak{p} z^{\mathfrak{p}} - \sum_{n=\mathfrak{p}+1}^{\infty} n a_n z^n} \right| \leq \frac{\sum_{n=\mathfrak{p}+j}^{\infty} n(n-\mathfrak{p}) |a_n| |z|^{n-\mathfrak{p}}}{\mathfrak{p} - \sum_{n=\mathfrak{p}+j}^{\infty} n |a_n| |z|^{n-\mathfrak{p}}} \leq \mathfrak{p} - \xi.$$

Therefore , we have

$$\sum_{n=p+j}^{\infty} \frac{n(n-\xi)}{\mathfrak{p}(\mathfrak{p}-\xi)} |a_n| |z|^{n-\mathfrak{p}} \leq 1. \qquad (20)$$

By Theorem 1

$$\sum_{n=\mathfrak{p}+j}^{\infty} \frac{1}{\Sigma(n)} |a_n| \leq 1. \qquad (21)$$

From (20) and (21) , will be true if $\frac{n(n-\xi)}{\mathfrak{p}(\mathfrak{p}-\xi)} |z|^{n-\mathfrak{p}} \leq \frac{1}{\Sigma(n)}$

$$|z| \leq \left\{ \left(\frac{\mathfrak{p}(\mathfrak{p}-\xi)}{n(n-\xi)} \right) \frac{1}{\Sigma(n)} \right\}^{\frac{1}{n-\mathfrak{p}}}.$$

Theorem 10: Let $h_1(z) = z^{\mathfrak{p}}$ and $h_n(z) = z^p - \Sigma(n) z^n$ for $n \geq 1 + \mathfrak{p}$. Then $h(z) \in \Gamma^*(\mathfrak{p}, \lambda, \phi, \delta, \alpha)$ if and only if $h(z) = \lambda_1 h_1(z) + \sum_{n=\mathfrak{p}+j}^{\infty} \lambda_n h_n(z)$, where $\lambda_n \geq 0$ and $\lambda_1 + \sum_{n=\mathfrak{p}+j}^{\infty} \lambda_n = 1$.

2. Conclusion

The class of analytic and *p*-valent functions with negative coefficients in the open unit disk is introduced. Also, the important geometric properties are presented like distortion theorems, extreme points, closure theorems, and inclusion properties. Also, we obtain radii of convexity, starlikeness and close-to-convexity for functions belonging to class.

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