Sumer Journal for Pure Science (ISSN: 2790-7031)
2nd International Scientific Conference on Pure and Medical Sciences/University of Sumer 2024

$L_p(\mathcal{D})$ Complex Approximation Using a Modified Szasz-Durrmeyer Operator

Mohammed Hadi Lafta [*1,2] **Eman Samir Bhaya** [3,b]

[1]*Department of Mathematics, College of Education for Pure Sciences, University of Babylon, Iraq.*

[2]*Department of Mathematics, College of Education, University of Sumer, Thiqar, Iraq*

[3]*AL- Zahraa University for Women, College of Education, Karbala*

[*] Email address of the Corresponding Author: mmohamnedhadi@gmail.com

Abstract:

Many papers about the approximation of analytic functions on a compact unit disk using Szasz- Durrmeyer operator were introduced. These papers are restricted to analytic functions. Here we generalize szasz operator by using the definition of $L_p(\mathcal{D})$ to get a best approximation to a function in quasi normed spaces. The upper bound theorem and Voronovskaja type result are presented. .

Keywords: Compact disks' exact order of approximation, complex Szasz-Durrmeyer operators, Voronovksaja type results and simultaneous approximation.

التقريب العقدي $L_p(\mathcal{D})$ باستخدام مؤثر شاش– دورماير المطور

محمد هادي لفته[1،2*] **, ايمان سمير بهيه**[3]

[1] قسم الرياضيات, كلية التربية للعلوم الصرفة, جامعة بابل, محافظة بابل, العراق

[2] قسم الرياضيات, كلية التربية, جامعة سومر , محافظة ذي قار , العراق

[3] قسم الرياضيات, كلية التربية, جامعة الزهراء للبنات, محافظة كربلاء , العراق

الخلاصة

قُدمَ العديد من البحوث حول تقريب الدوال التحليلية على قرص الوحدة المرصوص باستخدام عامل شاش-دورماير. تقتصر هذه البحوث على الدوال التحليلية. في هذا البحث قُمنا بتعميم مؤثر شاش باستخدام التعريف $L_p(\mathcal{D})$ للحصول على أفضل النتائج للدالة في الفضاءات المعيارية الكاذبة، مثل نظرية الحد الأعلى ونتيجة نوع فورنسكاجا.

الكلمات المفتاحية: الرتبة التامة للتقريب للأقراص المرصوصة، مؤثر شاش- دورماير العقدي، نتائج نوع فورنسكاجا و التقريب المتزامن.

1. Introduction

Concerning the uniform convergence of the complex Bernstein polynomials in the complex plane, Bernstein [1] showed that the complex Bernstein polynomials $B_\delta(f)(z) = \sum_{\ell=0}^{\delta}\binom{\delta}{\ell} z^\ell (1-z)^{\delta-\ell} f\left(\frac{\ell}{\delta}\right)$ converge to f in $\mathcal{D}$ if $f:\mathcal{H}\longrightarrow\mathbb{C}$ is analytic in the open set $\mathcal{H}\subset\mathbb{C}$, with $\mathcal{D}\subset\mathcal{H}$ (with $\mathcal{D}=\{z\in\mathbb{C}: |z|<1\}$ is the open unit disk).

Gal in [2] collected Voronovskja-type results with quantitative estimates for the complex Bernstein, complex q-Bernstein, complex Baskakov, complex Szasz (more precisely Favard-Szasz- Mirakjan), complex Bernstein-Kantorovich, complex Balazs-Szabados, and complex Stancu-Kantorovich operators attached to analytic functions on compact disks.

In Gal [3], [4], [5], [6] . Gal and Gupta [7], [8], [9] , Gal-Gupta-Mahmudov [10], Gupta and Verma [11] , Mazhar and Totik [12] , Walczak [13], Wood [14], Wright [15], the approximation properties of various complex variants for Durrmeyer-type operators were intensively studied.

Define

$$L_p(\mathcal{D}) = \left\{ f:\mathcal{D}\longrightarrow\mathbb{C} : \|f\|_{L_p(\mathcal{D})} = \left(\int_{\mathcal{D}} |f|^p\right)^{\frac{1}{p}} < \infty \right\},$$

where $0 < p < 1$.

Below we introduce a modification of the complex szasz- durrmeyer operators:

$$E_\delta(f)(z) = \sum_{\ell=0}^{\infty} B_{\delta,\ell}(z) f\left(\frac{\ell}{\delta}\right), \tag{1}$$

where $B_{\delta,\ell}(z) = \binom{\delta}{\ell} z^\ell (1-z)^{\delta-\ell}$ *and* $\delta\in\mathbb{N}$.

Let us define the stancu type generalization of the operator (1) as follows:

Sumer Journal for Pure Science (ISSN: 2790-7031)
2nd International Scientific Conference on Pure and Medical Sciences/University of Sumer 2024

$$EM_{\delta}(\mathfrak{f})(z) = \delta \sum_{\ell=0}^{\infty} B_{\delta,\ell}(z) \left(\int_0^{+\infty} B_{\delta,\ell}(y) \mathfrak{f}(y) dy \right). \quad (2)$$

The aim of this paper s to prove approximation results using the operator in equation (2) for functions in $\mathrm{L_p}(\mathcal{D})$. Also, the exact order of approximation and the Voronovskaja type result by this operator is obtained.

2. Auxiliary Results

First, we need now the following uxiliary results.

Lemma 2.1: Let $\mathfrak{f} \in \mathrm{L_p}(\mathcal{D})$, $\mathfrak{f} : \mathcal{D} \longrightarrow \mathbb{C}$ and there exists $F, C > 0$ such that $|\mathfrak{f}(w)| \leq Ce^{Fw}$. Denoting $\mathfrak{f}(z) = \sum_{\ell=0}^{\infty} d_\ell\, z^\ell, d_\ell \in \mathcal{R}^+$, we have $EM_\delta(\mathfrak{f})(z) = \sum_{\lambda=0}^{\infty} d_\lambda\, EM_\delta(e_\lambda)(z)\ \forall\ \delta > F$ where e_λ is a complex valued function on $\mathbb{C}$.

Proof: For all $\theta \in \mathbb{N}$ and $0 < r < 1$, let us define

$$\mathfrak{f}_\theta(z) = \sum_{\mu=0}^{\theta} d_\mu z^\mu \text{ if } |z| \leq r \text{ } and \text{ } \mathfrak{f}_\theta(z) = \mathfrak{f}(w) \text{ if } \mathrm{w} \in (r, +\infty).$$

Since $|\mathfrak{f}_\theta(z)| \leq \sum_{\mu=0}^{\infty} |d_\mu| . r^\mu := C_r, \forall\ |z| \leq r$ and $\theta \in \mathbb{N},\ \mathfrak{f} \in \mathrm{L_p}(\mathcal{D})$, from the hypothesis on $\mathfrak{f}$ it is clear that for all $\theta \in \mathbb{N}$ it follows $|\mathfrak{f}_\theta(z)| \leq C_{r,\mathcal{R}} e^{Fw}, \forall\ w \in [0, +\infty)$.

This implies that for each fixed $\theta, \delta \in \mathbb{N}, \delta > F$

$$EM_\delta(\mathfrak{f}_\theta)(z) = \sum_{\mu=0}^{\infty} d_\mu z^{2\mu} \binom{\delta}{\mu} (1-z)^{\delta-\mu} \left(\delta \int_0^{\infty} \binom{\delta}{\mu} y^\mu (1-y)^{\delta-\mu} e^{Fy} dy \right).$$

Then using β distribution for the integral above to get:

$$EM_\delta(\mathfrak{f}_\theta)(z) = \sum_{\mu=0}^{\infty} d_\mu z^{2\mu} \binom{\delta}{\mu} (1-z)^{\delta-\mu} . \frac{\mu!}{(\delta-\mu)!}.$$

Using the definition of $\mathrm{L_p}(\mathcal{D})$, we obtain

$$\|EM_\delta(\mathfrak{f}_\theta)(z)\|_{\mathrm{L_p}(\mathcal{D})} = \left\| \sum_{\mu=0}^{\infty} d_\mu z^{2\mu} \binom{\delta}{\mu} (1-z)^{\delta-\mu} . \frac{\mu!}{(\delta-\mu)!} \right\|_{\mathrm{L_p}(\mathcal{D})}$$

Sumer Journal for Pure Science (ISSN: 2790-7031)
2nd International Scientific Conference on Pure and Medical Sciences/University of Sumer 2024

$$\leq c(p)\sum_{\mu=0}^{\infty}\left\|d_{\mu}z^{2\mu}\binom{\delta}{\mu}(1-z)^{\delta-\mu}.\frac{\mu!}{(\delta-\mu)!}\right\|_{\mathrm{L_p}(\mathcal{D})}$$

$$=c(p)\sum_{\mu=0}^{\infty}\left(\int_{\mathcal{D}}\left|d_{\mu}z^{2\mu}\binom{\delta}{\mu}(1-z)^{\delta-\mu}\frac{\mu!}{(\delta-\mu)!}\right|^{p}dz\right)^{\frac{1}{p}}$$

$$=c(p)\sum_{\mu=0}^{\infty}|d_{\mu}|\binom{\delta}{\mu}\frac{\mu!}{(\delta-\mu)!}\left(\int_{\mathcal{D}}\left|z^{2\mu}(1-z)^{\delta-\mu}\right|^{p}dz\right)^{\frac{1}{p}}$$

$$\leq c(p)\sum_{\mu=0}^{\infty}|d_{\mu}|\binom{\delta}{\mu}\frac{\mu!}{(\delta-\mu)!}\left(\int_{\mathcal{D}}|z|^{2\mu p}(1+|z|)^{(\delta-\mu)p}dz\right)^{\frac{1}{p}}$$

$$\leq c(p)\sum_{\mu=0}^{\infty}|d_{\mu}|\binom{\delta}{\mu}\frac{\mu!}{(\delta-\mu)!}\left(\int_{\mathcal{D}}r^{2\mu p}(1+r)^{(\delta-\mu)p}dz\right)^{\frac{1}{p}}$$

$$\leq (p)C_{r,\mathcal{R}}\sum_{\mu=0}^{\infty}r^{\mu}\binom{\delta}{\mu}(1+r)^{\delta-\mu}\frac{\mu!}{(\delta-\mu)!}\left(\int_{\mathcal{D}}dz\right)^{\frac{1}{p}}$$

$$=(p)C_{r,\mathcal{R}}\sum_{\mu=0}^{\infty}\binom{\delta}{\mu}\frac{\mu!}{(\delta-\mu)!}r^{\mu}(1+r)^{\delta-\mu}(\pi r^{2})^{\frac{1}{p}}<\infty,$$

We note that the final series is convergent according to the ratio criterion. Therefore $EM_{\delta}(\mathfrak{f}_{\theta})(z)$ is well-defined.

Denoting $\mathfrak{f}_{\theta,\lambda}(z)=d_{\lambda}(e_{\lambda})(z)$ if $|z|\leq r$ and $\mathfrak{f}_{\theta,\lambda}(w)=\frac{\mathfrak{f}(w)}{\theta+1}$ if w $\in(r,\infty)$.

Let $\mathfrak{f}_{\theta}(z)=\sum_{\lambda=0}^{\theta}\mathfrak{f}_{\theta,\lambda}(z)$. Because EM_{δ} is linear, we have

$$EM_{\delta}(\mathfrak{f}_{\theta})(z)=\sum_{\lambda=0}^{\theta}d_{\lambda}EM_{\delta}(e_{\lambda})(z),\forall\,|z|\leq r,$$

it's enough to show that $\lim_{\theta\to\infty}EM_{\delta}(\mathfrak{f}_{\theta})(z)=EM_{\delta}(\mathfrak{f})(z)$ for all fixed $\delta\in\mathbb{N}$ and $|z|\leq r$. As $\lim_{\theta\to\infty}\|\mathfrak{f}_{\theta}-\mathfrak{f}\|_{\mathrm{L_p}(\mathcal{D})}$=0, and

$$\|EM_{\delta}(\mathfrak{f}_{\theta})(z)-EM_{\delta}(\mathfrak{f})(z)\|_{\mathrm{L_p}(\mathcal{D})}\leq c(p,\delta)\|\mathfrak{f}_{\theta}-\mathfrak{f}\|_{\mathrm{L_p}(\mathcal{D})},$$

Sumer Journal for Pure Science (ISSN: 2790-7031)
2nd International Scientific Conference on Pure and Medical Sciences/University of Sumer 2024

for any $|z| \leq r$, The proof is finished ■

Lemma 2.2: If $(e_\lambda)(z) = z^\lambda$ and $S_{\delta,\lambda}(z) = EM_\delta(e_\lambda)(z)$, then

$$S'_{\delta,\lambda}(z) = \frac{\delta-\ell}{1-z} S_{\delta,\lambda+1}(z) - \left(\frac{\lambda+1}{1-z} - \frac{\ell}{z}\right) S_{\delta,\lambda}(z).$$

Proof: We have

$$S_{\delta,\lambda}(z) = \delta \sum_{\ell=0}^{\infty} B_{\delta,\ell}(z) \left(\int_0^{+\infty} B_{\delta,\ell}(y) y^\lambda dy \right)$$

$$= \delta \sum_{\ell=0}^{\infty} B_{\delta,\ell}(z) \left(\int_0^{+\infty} \binom{\delta}{\ell} (1-y)^{\delta-\ell} y^{\ell+\lambda} dy \right).$$

Then using β distribution for the integral above to get:

$$S_{\delta,\lambda}(z) = \delta\delta! \sum_{\ell=0}^{\infty} B_{\delta,\ell}(z) \frac{(\ell+\lambda)!}{\ell!\,(\delta+\lambda+1)!}. \tag{3}$$

This immediately implies

$$S'_{\delta,\lambda}(z) = \delta\delta! \sum_{\ell=0}^{\infty} \frac{(\ell+\lambda)!}{\ell!\,(\delta+\lambda+1)!} \left(\binom{\delta}{\ell} z^\ell (1-z)^{\delta-\ell} \right)'$$

$$= \delta\delta! \sum_{\ell=0}^{\infty} \frac{(\ell+\lambda)!}{\ell!\,(\delta+\lambda+1)!} \left(\binom{\delta}{\ell} \ell z^{\ell-1} \cdot (1-z)^{\delta-\ell} + \binom{\delta}{\ell} z^\ell \cdot \delta - \ell(1-z)^{\delta-\ell-1} \right)$$

$$= \frac{\ell}{z} S_{\delta,\lambda}(z) + \frac{\delta-\ell}{1-z} \delta\delta! \sum_{\ell=0}^{\infty} \frac{(\ell+\lambda+1)!}{\ell!\,(\delta+\lambda+2)!} \cdot \binom{\delta}{\ell} z^\ell \cdot (1-z)^{\delta-\ell} \cdot \frac{\delta+\lambda+2}{\ell+\lambda+1}$$

$$= \frac{\ell}{z} S_{\delta,\lambda}(z) + \frac{\delta-\ell}{1-z} S_{\delta,\lambda+1}(z) - \frac{\delta-\ell}{1-z} \cdot \frac{\lambda+1}{\delta-\ell} \delta\delta! \sum_{\ell=0}^{\infty} \frac{(\ell+\lambda+1)!}{\ell!\,(\delta+\lambda+1)!} \cdot \binom{\delta}{\ell} z^\ell \cdot (1-z)^{\delta-\ell} \cdot \frac{1}{\ell+\lambda+1}$$

$$= \frac{\ell}{z} S_{\delta,\lambda}(z) + \frac{\delta-\ell}{1-z} S_{\delta,\lambda+1}(z) - \frac{\lambda+1}{1-z} S_{\delta,\lambda}(z),$$

which shows the statement's recurrence ■

Corollary 2.3: By direct calculations we get if

$$S'_{\delta,\lambda}(z) = \frac{\delta - \ell}{1 - z} S_{\delta,\lambda+1}(z) - \left(\frac{\lambda + 1}{1 - z} - \frac{\ell}{z}\right) S_{\delta,\lambda}(z),$$

then

$$S_{\delta,\lambda+1}(z) = \frac{1 - z}{\delta - \ell} S'_{\delta,\lambda}(z) + \left(\frac{z(\lambda + \ell + 1) - \ell}{z(\delta - \ell)}\right) S_{\delta,\lambda}(z).$$

3. Main Results

We first introduce the following upper bound theorem.

Theorem 3.1: Let $\mathcal{D} = \{z \in \mathbb{C}: |z| < 1\}$ and $f \in L_p(\mathcal{D})$, $f : \mathcal{D} \longrightarrow \mathbb{C}$, and $f(z) = \sum_{\lambda=0}^{\infty} d_\lambda z^\lambda$. Then for $\delta > l$ and $\delta, \ell \in \mathbb{N}$,

$$\|EM_\delta(f)(z) - f(z)\|_{L_p(\mathcal{D})} \leq$$

$$\frac{2^{\frac{1}{p}-1}}{\delta(\delta - \ell)} \sum_{\lambda=0}^{\infty} |d_\lambda| \left[\delta\big((1 + r)\tau + 3\big) + \ell + 1\right] (\pi r^2)^{\frac{1}{p}} \lambda (2r)^{\lambda-1} < \infty.$$

Proof: By using definition of $L_p(\mathcal{D})$, we get

$$\|EM_\delta(f) - f\|_{L_p(\mathcal{D})} = \left(\int_{\mathcal{D}} |EM_\delta(f) - f|^p dz\right)^{\frac{1}{p}},$$

and by using the recurrence relation of corollary 2.3, we have

$$S_{\delta,\lambda+1}(z) = \frac{1 - z}{\delta - \ell} S'_{\delta,\lambda}(z) + \left(\frac{z(\lambda + \ell + 1) - \ell}{z(\delta - \ell)}\right) S_{\delta,\lambda}(z)$$

for all $\lambda \in \{0, 1, 2, \dots\}, \delta \in \mathbb{N}$. This gives us the recurrence formula immediately

$$\left\|S_{\delta,\lambda}(z) - z^\lambda\right\|_{L_p(\mathcal{D})} = \left\|\frac{1 - z}{\delta - \ell}\left[S_{\delta,\lambda-1}(z) - z^{\lambda-1}\right]' + \left(\frac{z(\lambda + \ell + 1) - \ell}{z(\delta - \ell)}\right)\left[S_{\delta,\lambda-1}(z) - z^{\lambda-1}\right] + \frac{2\lambda - 1}{\delta} z^{\lambda-1}\right\|_{L_p(\mathcal{D})}$$

$$\le 2^{\frac{1}{p}-1}\left[\left\|\frac{1-z}{\delta-\ell}[S_{\delta,\lambda-1}(z)-z^{\lambda-1}]'\right\|_{L_p(\mathcal{D})}\right.$$

$$+\left\|\left(\frac{z(\lambda+\ell+1)-\ell}{z(\delta-\ell)}\right)[S_{\delta,\lambda-1}(z)-z^{\lambda-1}]\right\|_{L_p(\mathcal{D})}$$

$$\left.+\left\|\frac{2\lambda-1}{\delta}z^{\lambda-1}\right\|_{L_p(\mathcal{D})}\right]$$

$$=2^{\frac{1}{p}-1}\left[\left(\int_{\mathcal{D}}\left|\frac{1-z}{\delta-\ell}[S_{\delta,\lambda-1}(z)-z^{\lambda-1}]'\right|^p dz\right)^{\frac{1}{p}}\right.$$

$$\left.+\left(\int_{\mathcal{D}}\left|\left(\frac{z(\lambda+\ell+1)-\ell}{z(\delta-\ell)}\right)[S_{\delta,\lambda-1}(z)-z^{\lambda-1}]\right|^p dz\right)^{\frac{1}{p}}+\left(\int_{\mathcal{D}}\left|\frac{2\lambda-1}{\delta}z^{\lambda-1}\right|^p dz\right)^{\frac{1}{p}}\right],$$

$$\le 2^{\frac{1}{p}-1}\left[\frac{1+r}{\delta-\ell}\left(\int_{\mathcal{D}}\left|[S_{\delta,\lambda-1}(z)-z^{\lambda-1}]'\right|^p dz\right)^{\frac{1}{p}}\right.$$

$$\left.+\left(\frac{r(\lambda+\ell+1)-\ell}{r(\delta-\ell)}\right)\left(\int_{\mathcal{D}}\left|[S_{\delta,\lambda-1}(z)-z^{\lambda-1}]\right|^p dz\right)^{\frac{1}{p}}+\frac{2\lambda-1}{\delta}\left(\int_{\mathcal{D}}\left|z^{\lambda-1}\right|^p dz\right)^{\frac{1}{p}}\right],$$

hence

$$\left\|S_{\delta,\lambda}(z)-z^{\lambda}\right\|_{L_p(\mathcal{D})}\le 2^{\frac{1}{p}-1}\left[\frac{1+r}{\delta-\ell}\left\|[S_{\delta,\lambda-1}(z)-z^{\lambda-1}]'\right\|_{L_p(\mathcal{D})}\right.$$

$$\left.+\left(\frac{r(\lambda+\ell+1)-\ell}{r(\delta-\ell)}\right)\left\|S_{\delta,\lambda-1}(z)-z^{\lambda-1}\right\|_{L_p(\mathcal{D})}+\frac{2\lambda-1}{\delta}\left\|z^{\lambda-1}\right\|_{L_p(\mathcal{D})}\right].$$

Then using Bernstein's inequality for $S_{\delta,\lambda}(z)$ polynomial of the degree $\lambda-1$ we obtain:

$$\left\|[S_{\delta,\lambda-1}(z)-z^{\lambda-1}]'\right\|_{L_p(\mathcal{D})}\le c(\lambda,p)\left\|S_{\delta,\lambda-1}(z)-z^{\lambda-1}\right\|_{L_p(\mathcal{D})},$$

where $c(\lambda,p)$ is a positive constant depending on λ and p.

Therefore, it follows

Sumer Journal for Pure Science (ISSN: 2790-7031)
2nd International Scientific Conference on Pure and Medical Sciences/University of Sumer 2024

$$\left\|S_{\delta,\lambda}(z)-z^{\lambda}\right\|_{L_p(\mathcal{D})}\le 2^{\frac{1}{p}-1}\left[\frac{(1+r)c(\lambda,p)}{\delta-\ell}\left\|S_{\delta,\lambda-1}(z)-z^{\lambda-1}\right\|_{L_p(\mathcal{D})}\right.$$

$$\left.+\left(\frac{r(\lambda+\ell+1)-\ell}{r(\delta-\ell)}\right)\left\|S_{\delta,\lambda-1}(z)-z^{\lambda-1}\right\|_{L_p(\mathcal{D})}+\frac{2\lambda-1}{\delta}\left\|z^{\lambda-1}\right\|_{L_p(\mathcal{D})}\right],$$

consequently, we get

$$\left\|S_{\delta,\lambda}(z)-z^{\lambda}\right\|_{L_p(\mathcal{D})}\le 2^{\frac{1}{p}-1}\left[\frac{(1+r)c(\lambda,p)}{\delta-\ell}\left(\int_{\mathcal{D}}\left|S_{\delta,\lambda-1}(z)-z^{\lambda-1}\right|^{p}dz\right)^{\frac{1}{p}}\right.$$

$$\left.+\left(\frac{r(\lambda+\ell+1)-\ell}{r(\delta-\ell)}\right)\left(\int_{\mathcal{D}}\left|S_{\delta,\lambda-1}(z)-z^{\lambda-1}\right|^{p}dz\right)^{\frac{1}{p}}+\frac{2\lambda-1}{\delta}\left(\int_{\mathcal{D}}\left|z^{\lambda-1}\right|^{p}dz\right)^{\frac{1}{p}}\right],$$

Using equation (3) and after performing calculations for any $> l$, we get

$$\left\|S_{\delta,\lambda}(z)-z^{\lambda}\right\|_{L_p(\mathcal{D})}\le\frac{2^{\frac{1}{p}-1}}{\delta(\delta-\ell)}\left[\delta\big((1+r)c(\lambda,p)+3\big)+\ell+1\right]\lambda(2r)^{\lambda-1}(\pi r^{2})^{\frac{1}{p}}.$$

Now, by Lemma 2.1, we can write

$$EM_{\delta}(f)(z)=\sum_{\lambda=0}^{\infty}d_{\lambda}S_{\delta,\lambda}(z)$$

which implies

$$\|EM_{\delta}(f)(z)-f(z)\|_{L_p(\mathcal{D})}=\left\|\sum_{\lambda=0}^{\infty}d_{\lambda}S_{\delta,\lambda}(z)-z^{\lambda}\right\|_{L_p(\mathcal{D})}$$

$$\le\sum_{\lambda=0}^{\infty}|d_{\lambda}|\left\|S_{\delta,\lambda}(z)-z^{\lambda}\right\|_{L_p(\mathcal{D})}$$

$$\le\frac{2^{\frac{1}{p}-1}}{\delta(\delta-\ell)}\sum_{\lambda=0}^{\infty}|d_{\lambda}|\left[\delta\big((1+r)c(\lambda,p)+3\big)+\ell+1\right]\lambda(2r)^{\lambda-1}(\pi r^{2})^{\frac{1}{p}}.$$

Then since $0<r<1$, so

$$\|EM_{\delta}(f)(z)-f(z)\|_{L_p(\mathcal{D})}\le$$

$$\frac{2^{\frac{1}{p}-1}}{\delta(\delta-\ell)}\sum_{\lambda=0}^{\infty}|d_\lambda|\left[\delta\big((1+r)c(\lambda,p)+3\big)+\ell+1\right](\pi r^2)^{\frac{1}{p}}\lambda(2r)^{\lambda-1}<\infty.$$

The Voronovskja type result that holds

Theorem 3.2: Let $\mathcal{D}=\{z\in\mathbb{C}:|z|<1\}$ and $f\in L_p(\mathcal{D})$, $f:\mathcal{D}\longrightarrow\mathbb{C}$, that is $f(z)=\sum_{\lambda=0}^{\infty}d_\lambda z^\lambda$. Then for all $|z|\le r$, $\delta>l$ and $\delta,\ell\in\mathbb{N}$

$$\left\|EM_\delta(f)(z)-f(z)-\frac{1}{\delta}f'(z)-\frac{z}{\delta}f''(z)\right\|_{L_p(\mathcal{D})}\le$$

$$\frac{(2\pi r^2)^{\frac{1}{p}}c(\lambda,p)}{\delta(\delta-\ell)}\sum_{\lambda=1}^{\infty}|d_\lambda|[\delta(1+r)+2(\delta+\ell)+\lambda(\delta-\ell)+1](\lambda-1)(2r)^{\lambda-2}<\infty.$$

Proof: We denote $e_\lambda(z)=z^\lambda,\lambda=0,1,2,\ldots$ and $S_{\delta,\lambda}(z)=EM_\delta\big(e_{\lambda,}z\big)$. By Lemma 2.1, we can write $EM_\delta(f)(z)=\sum_{\lambda=0}^{\infty}d_\lambda S_{\delta,\lambda}(z)$ for all $\delta\in\mathbb{N}$. Also

$$\frac{z\,f''(z)+f'(z)}{\delta}=\frac{z}{\delta}\sum_{\lambda=2}^{\infty}d_\lambda\lambda(\lambda-1)z^{\lambda-2}+\frac{1}{\delta}\sum_{\lambda=1}^{\infty}d_\lambda\lambda z^{\lambda-1}$$

$$=\frac{1}{\delta}\sum_{\lambda=1}^{\infty}d_\lambda[\lambda(\lambda-1)+\lambda]z^{\lambda-1}.$$

Thus

$$EM_\delta(f)(z)-f(z)-\frac{1}{\delta}f'(z)-\frac{z}{\delta}f''(z)=\sum_{\lambda=0}^{\infty}d_\lambda\left(S_{\delta,\lambda}(z)-e_\lambda(z)-\frac{\lambda^2z^{\lambda-1}}{\delta}\right),$$

for any $\delta\in\mathbb{N},z\in\mathcal{D}_{\mathcal{R}}$.

By corollary 2.3, for any $\delta\in\mathbb{N}$ and $\lambda=0,1,2,\ldots$, we have

$$S_{\delta,\lambda+1}(z)=\frac{1-z}{\delta-\ell}S'_{\delta,\lambda}(z)+\left(\frac{z(\lambda+\ell+1)-\ell}{z(\delta-\ell)}\right)S_{\delta,\lambda}(z).$$

If we denote

$$B_{\lambda,\delta}(z)=S_{\delta,\lambda}(z)-e_\lambda(z)-\frac{\lambda^2z^{\lambda-1}}{\delta},\tag{4}$$

Sumer Journal for Pure Science (ISSN: 2790-7031)
2nd International Scientific Conference on Pure and Medical Sciences/University of Sumer 2024

then it's clear that $B_{\lambda,\delta}(z)$ is a polynomial with degree less than or equal to λ. Using the aforementioned recurrence relation and some easy calculation obtain:

$$B_{\lambda,\delta}(z) = \frac{1-z}{\delta-\ell} B'_{\lambda-1,\delta}(z) + \left(\frac{z(\lambda+\ell)-\ell}{z(\delta-\ell)}\right) B_{\lambda-1,\delta}(z) + \frac{2z^{\lambda-2}(\lambda-1)^3}{\delta^2}$$

for any $\lambda \geq 1, \delta, \ell \in \mathbb{N}, \delta > l, \&\ |z| \leq r$.

Hence

$$\|B_{\lambda,\delta}(z)\|_{L_p(\mathcal{D})}$$

$$= \left\|\frac{1-z}{\delta-\ell} B'_{\lambda-1,\delta}(z) + \left(\frac{z(\lambda+\ell)-\ell}{z(\delta-\ell)}\right) B_{\lambda-1,\delta}(z) + \frac{2z^{\lambda-2}(\lambda-1)^3}{\delta^2}\right\|_{L_p(\mathcal{D})}$$

$$\leq 2^{\frac{1}{p}-1}\left[\left\|\frac{1-z}{\delta-\ell} B'_{\lambda-1,\delta}(z)\right\|_{L_p(\mathcal{D})} + \left\|\left(\frac{z(\lambda+\ell)-\ell}{z(\delta-\ell)}\right) B_{\lambda-1,\delta}(z)\right\|_{L_p(\mathcal{D})}\right.$$

$$\left. + \left\|\frac{2z^{\lambda-2}(\lambda-1)^3}{\delta^2}\right\|_{L_p(\mathcal{D})}\right].$$

This implies

$$\|B_{\lambda,\delta}(z)\|_{L_p(\mathcal{D})}$$

$$\leq 2^{\frac{1}{p}-1}\left[\frac{1+r}{\delta-\ell}\|B'_{\lambda-1,\delta}(z)\|_{L_p(\mathcal{D})} + \frac{2(\lambda-1)^3}{\delta^2}\|z^{\lambda-2}\|_{L_p(\mathcal{D})}\right]$$

$$+2^{\frac{1}{p}-1}\left[\frac{r(\lambda+\ell)-\ell}{r(\delta-\ell)}\|B_{\lambda-1,\delta}(z)\|_{L_p(\mathcal{D})}\right].$$

Now from the Bernstein inequality we obtain

$$\|B'_{\lambda-1,\delta}(z)\|_{L_p(\mathcal{D})} \leq c(\lambda,p)\|B_{\lambda-1,\delta}(z)\|_{L_p(\mathcal{D})},$$

where $c(\lambda, p)$ is a positive constant depending on λ and p.

From (4), we have

Sumer Journal for Pure Science (ISSN: 2790-7031)
2nd International Scientific Conference on Pure and Medical Sciences/University of Sumer 2024

$$\left\|B'_{\lambda-1,\delta}(z)\right\|_{L_p(\mathcal{D})} \le c(\lambda,p)\left\|S_{\delta,\lambda-1}(z) - z^{\lambda-1} - \frac{(\lambda-1)^2 z^{\lambda-2}}{\delta}\right\|_{L_p(\mathcal{D})}$$

$$\le 2^{\frac{1}{p}-1}c(\lambda,p)\left\|S_{\delta,\lambda-1}(z) - z^{\lambda-1}\right\|_{L_p(\mathcal{D})} + 2^{\frac{1}{p}-1}c(\lambda,p)\left\|\frac{(\lambda-1)^2 z^{\lambda-2}}{\delta}\right\|_{L_p(\mathcal{D})}$$

$$\le \frac{2^{\frac{1}{p}-1}c(\lambda,p)}{\delta(\delta-\ell)}[\delta(1+r)+2(\delta+\ell)+\lambda(\delta-\ell)+1](\lambda-1)(2r)^{\lambda-2}(\pi r^2)^{\frac{1}{p}}.$$

Hence

$$\left\|B_{\lambda,\delta}(z)\right\|_{L_p(\mathcal{D})} \le 2^{\frac{1}{p}-1}\left[\frac{r(\lambda+\ell)-\ell}{r(\delta-\ell)}\left\|B_{\lambda-1,\delta}(z)\right\|_{L_p(\mathcal{D})}\right]$$

$$+\frac{2^{\frac{1}{p}-1}c(\lambda,p)}{\delta(\delta-\ell)}[\delta(1+r)+2(\delta+\ell)+\lambda(\delta-\ell)+1](\lambda-1)(2r)^{\lambda-2}(\pi r^2)^{\frac{1}{p}}.$$

Because it is the case that for $\lambda = 1$ we obtain $B_{\lambda,\delta}(z) = 0$, for $\lambda \ge 2$ in the latter relation, we obtain that by using the same calculations as in [16] page 117, we obtain that

$$B_{\lambda,\delta}(z) \le \frac{(2\pi r^2)^{\frac{1}{p}}c(\lambda,p)}{\delta(\delta-\ell)}[\delta(1+r)+2(\delta+\ell)+\lambda(\delta-\ell)+1](\lambda-1)(2r)^{\lambda-2}. \qquad (5)$$

Since $EM_\delta(f)(z) - f(z) - \frac{1}{\delta}f'(z) - \frac{z}{\delta}f''(z) = \sum_{\lambda=0}^{\infty} d_\lambda B_{\lambda,\delta}(z)$

$$\left\|EM_\delta(f)(z) - f(z) - \frac{1}{\delta}f'(z) - \frac{z}{\delta}f''(z)\right\|_{L_p(\mathcal{D})} = \left\|\sum_{\lambda=0}^{\infty} d_\lambda B_{\lambda,\delta}(z)\right\|_{L_p(\mathcal{D})}$$

$$\le \sum_{\lambda=0}^{\infty}|d_\lambda|\left\|B_{\lambda,\delta}(z)\right\|_{L_p(\mathcal{D})},$$

by (5) we reach the desired result. ■

4. Conclusions

In our paper, the definition of $L_p(\mathcal{D})$ was presented to study the approximation properties. We discuss the approximation properties by using generalize szasz operator by using the definition of $L_p(\mathcal{D})$. We obtained some approximate properties, like, upper estimate, voronovskaya type and exact estimate formula.

References

[1] G. G. Lorentz, "Bernstein Polynomials," *American Mathematical Soc.,* 2012.

[2] S. G. Gal, "Approximation by complex Bernstein and convolution type Operators," *World Scientific,* vol. 8, pp. 269- 279, 2009.

[3] S. G. Gal, "Approximation and geometric properties of complex Favard-Szasz-Mirakjan operators in compact disks," *Computers & Mathematics with Applications*, vol. 56, no. 4, p. 1121–1127, 2008.

[4] S. G. Gal, "Approximation by complex Bernstein-Durrmeyer polynomials with Jacobi weights in compact disks," *Mathematica Balkanica (NS),* vol. 24, no. 2, p. 103–119, 2010.

[5] S. G. Gal, "Approximation by complex genuine Durrmeyer type polynomials in compact disks," *Applied Mathematics and Computation,* vol. 217, no. 5, p. 1913–1920, 2010.

[6] S. G. Gal, "Approximation of analytic functions without exponential growth conditions by complex Favard Szasz-Mirakjan operators," *Rendiconti del Circolo Matematico di Palermo,* vol. 59, no. 3, p. 367–376, 2010.

[7] S. G. G. a. V. Gupta, "Approximation by complex Szasz- Durrmeyer Operators in compact disks," *Acta Mathematica Scientia*, vol. 34B, no. 4, pp. 1157- 1165, 2014.

[8] S. G. G. a. V. Gupta, " Approximation by a Durrmeyer-type operator in compact disks," *Annali dell Universita di Ferrara*, vol. 57, no. 2, p. 261–274, 2011.

[9] S. G. G. a. V. Gupta, " Quantitative estimates for a new complex Durrmeyer operator in compact disks," *Appl Math Comput*, vol. 218, no. 6, p. 2944–2951, 2011.

[10] V. G. a. N. I. M. S. G. Gal, " Approximation by a complex q-Durrmeyer type operator," *Ann Univ Ferrara Sez VII Sci Mat*, vol. 58, no. 1, p. 65–87, 2012.

[11] V. G. a. D. K. Verma, " Approximation by complex Favard-Szasz-Mirakjan-Stancu operators in compact disks," *Mathematical Sciences*, vol. 6, no. 1, pp. 1-8, 2012.

[12] S. M. M. a. V. Totik, "Approximation by modified Szasz operators," *Acta Sci Math*, vol. 49, no. 4, p. 257–269, 1985..

[13]]. Z. Walczak, "On modified Szasz–Mirakyan operators," *Novi Sad Journal of Mathematics*, vol. 33, no. 1, p. 93–107, 2003 .

[14] B. Wood, " Generalized Szasz operators for the approximation in the complex domain," *SIAM J. Appl. Math*, vol. 17, no. 4, p. 790–801, 1969.

[15] E. M. Wright, " The Bernstein approximation polynomials in the complex plane," *J. Lond. Math. Soc.,* vol. 1, no. 4, pp. 265-269, 1930.

[16] F. D. a. W. P. J. J. Gergen, " Convergence of extended Bernstein polynomials in the complex plane," *Pacific J. Math.,* vol. 13, no. 4, pp. 1171-1180, 1963.